The Wilson-Wilson Experiment

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(July 30, 2008)

1 Problem

An example of unipolar induction[1] is a slab of material with (relative) permittivity ϵ_r and (relative) permeability μ_r that is immersed in a uniform, time-independent external magnetic field $\mathbf{H_0}$ parallel to the plane of the slab, such that if the slab has velocity $\mathbf{v}$ perpendicular to $\mathbf{H}_0$ but also in the plane of the slab, a nonzero voltage difference V is detected between sliding contacts (fixed in the laboratory) on opposite sides of the slab.

Deduce this voltage difference. Consider also the case that the material is in the form of a (long) cylindrical shell, of length l and radii $a < b$, that rotates with constant angular velocity ω about its axis, which is parallel to the external field $\mathbf{H}_0$. You may assume that all velocities are small compared to the speed of light c.

Experiments with rotating magnetized cylinders were first performed by Faraday [2, 3]. The variant with a permeable, dielectric cylindrical shell and an external magnetic field was studied in 1913 by Wilson and Wilson [4] as a test of special relativity;[2] see also [6].

2 Solution

A theme of this example is to what extent the electrodynamics of systems involving only low velocities are nonetheless "relativistic", and whether rotating systems can be successfully analyzed using the methods of special relativity.

We work in Gaussian units, where the speed of light appears in Maxwell's equations in a manner that alerts us to possible relativistic effects.[3]

2.1 Slab with Linear Motion

We first consider the example of a slab that moves with constant velocity with respect to the lab frame (which we assume to be an inertial frame), where it is clear that special relativity can be used to describe the electrodynamics in the rest frame of the slab. Can we, however, give an analysis in the lab frame with no explicit mention of the theory of relativity, other than that which is implicit in "Maxwell's equations"?

---

[1] The term unipolar induction was coined by Weber (1840) [1].

[2] In 1904, Wilson [5] studied a rotating dielectric with unit (relative) permeability, which did not test the difference between eqs. (6) and (18) below.

[3] A textbook with extensive discussion of the electrodynamics of moving dielectric and magnetic media is [7], especially Chap. E III.

2.1.1 A Naïve analysis

For example, since the slab has permittivity ϵ different from unity, it is a polarizable medium. Hence, there exist bound charges within the medium that can move relative to the bulk in response to a macroscopic electric field $\mathbf{E}$ to create a density of polarization $\mathbf{P}$ according to,

$$\mathbf{P} = \frac{\mathbf{D} - \mathbf{E}}{4\pi} = \frac{\epsilon_r - 1}{4\pi}\mathbf{E}, \tag{1}$$

in the approximation that the medium is linear and isotropic, such that $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon_r\mathbf{E}$. However, this simple interpretation of the permittivity ϵ_r can only be made in the rest frame of the slab.

If we ignore this limitation, we might argue that the polarizable charges in the present example have velocity $\mathbf{v}$, and that they are in a magnetic field whose strength inside the medium is $\mathbf{B} = \mu_r\mathbf{H}_0$, so that the Lorentz force (a part of the larger notion of "Maxwell's equations") on an electric charge e is,

$$e\frac{\mathbf{v}}{c} \times \mathbf{B} = e\mathbf{E}_{\text{Lorentz}}, \tag{2}$$

where,

$$\mathbf{E}_{\text{Lorentz}} = \frac{\mathbf{v}}{c} \times \mathbf{B} = \mu_r\frac{\mathbf{v}}{c} \times \mathbf{H}_0. \tag{3}$$

This suggests that effective field on charges in the medium is $\mathbf{E}+\mathbf{E}_{\text{Lorentz}}$, and the polarization $\mathbf{P}$ of the medium related by,

$$\mathbf{P} = \frac{\epsilon_r - 1}{4\pi}(\mathbf{E} + \mathbf{E}_{\text{Lorentz}}). \tag{4}$$

Arguments of this sort predate the theory of relativity, and are often used to explain unipolar induction in examples such as Faraday's disk where $\epsilon_r = 1 = \mu_r$ for all relevant media [7]-[11].

Since there are no free charges in this example, the electric displacement $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ vanishes.[4] Thus, the electric field $\mathbf{E}$ within the medium is given by $\mathbf{E} = -4\pi\mathbf{P}$, and so,

$$\mathbf{E} = -\left(1 - \frac{1}{\epsilon_r}\right)\mathbf{E}_{\text{Lorentz}} = -\mu_r\left(1 - \frac{1}{\epsilon_r}\right)\frac{\mathbf{v}}{c} \times \mathbf{H}_0. \tag{5}$$

Finally, the voltage observed between the sliding contacts would be,

$$V = -Ed = \mu_r\left(1 - \frac{1}{\epsilon_r}\right)\frac{vH_0d}{c}, \tag{6}$$

where d is the thickness of the slab.[5]

[4] As there are no free charges, $\boldsymbol{\nabla} \cdot \mathbf{D} = 0$, and in this stationary example, $\boldsymbol{\nabla} \times \mathbf{D} = 0$, so that $\mathbf{D} = 0$ according to Helmholtz' theorem.

[5] The result (6), but not necessarily the above derivation, is attributed by Wilson and Wilson [4] to a pre-relativistic theory of H.A. Lorentz.

2.1.2 A Special-Relativistic Analysis

We try to confirm the preceding analysis using the theory of special relativity.[6]

We know that the form of Maxwell's equations is the same in all inertia frames. However, we can only say with confidence that the so-called constitutive equations (for a linear, isotropic medium),

$$\mathbf{D}^\star = \epsilon_r \mathbf{E}^\star, \tag{7}$$

$$\mathbf{B}^\star = \mu_r \mathbf{H}^\star, \tag{8}$$

and (for a medium with electrical conductivity σ),

$$\mathbf{J}^\star = \sigma \mathbf{E}^\star, \tag{9}$$

hold in an inertial rest frame of the medium, where the fields are denoted with a *. That is, our understanding of electrical media is not fully relativistic in that there is a preferred frame (the rest frame of the medium) in which key physical parameters are given a special meaning. Hence, any analysis of electrical media involves some level of conflict with the spirit of special relativity.

The Lorentz transformations of the fields from the lab frame to the rest frame are [7],

$$\mathbf{E}^\star_\parallel = \mathbf{E}_\parallel, \qquad \mathbf{E}^\star_\perp = \gamma\left(\mathbf{E}_\perp + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \qquad \mathbf{B}^\star_\parallel = \mathbf{B}_\parallel, \qquad \mathbf{B}^\star_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v}}{c}\times\mathbf{E}\right), \tag{10}$$

$$\mathbf{D}^\star_\parallel = \mathbf{D}_\parallel, \qquad \mathbf{D}^\star_\perp = \gamma\left(\mathbf{D}_\perp + \frac{\mathbf{v}}{c}\times\mathbf{H}\right), \qquad \mathbf{H}^\star_\parallel = \mathbf{H}_\parallel, \qquad \mathbf{H}^\star_\perp = \gamma\left(\mathbf{H}_\perp - \frac{\mathbf{v}}{c}\times\mathbf{D}\right), \tag{11}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

The constitutive equations (7) and (8) can now be expressed in terms of lab-frame quantities (for any value of γ) as,

$$\mathbf{D} + \frac{\mathbf{v}}{c}\times\mathbf{H} = \epsilon_r\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \qquad \mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E} = \mu_r\left(\mathbf{H} - \frac{\mathbf{v}}{c}\times\mathbf{D}\right). \tag{12}$$

Inserting these expressions into one another, we obtain,[7]

$$\mathbf{D} - \epsilon_r\mu_r\frac{v^2}{c^2}\mathbf{D}_\perp = \epsilon_r\left(\mathbf{E} - \frac{v^2}{c^2}\mathbf{E}_\perp\right) + (\epsilon_r\mu_r - 1)\frac{\mathbf{v}}{c}\times\mathbf{H}, \tag{13}$$

$$\mathbf{B} - \epsilon_r\mu_r\frac{v^2}{c^2}\mathbf{B}_\perp = \mu_r\left(\mathbf{H} - \frac{v^2}{c^2}\mathbf{H}_\perp\right) - (\epsilon_r\mu_r - 1)\frac{\mathbf{v}}{c}\times\mathbf{E}. \tag{14}$$

Because $v \ll c$ in the present example it suffices to take the low-velocity limit that terms in v^2/c^2 are negligible while keeping terms in v/c.[8] Thus, the constitutive equations of an electrical medium that has velocity $\mathbf{v} \ll c$ in the lab frame are,

$$\mathbf{D} = \epsilon_r\mathbf{E} + (\epsilon_r\mu_r - 1)\frac{\mathbf{v}}{c}\times\mathbf{H}, \tag{15}$$

$$\mathbf{B} = \mu_r\mathbf{H} - (\epsilon_r\mu_r - 1)\frac{\mathbf{v}}{c}\times\mathbf{E}, \tag{16}$$

[6] This section is abstracted from [12].

[7] The lab-frame constitutive equations (13)-(14) were first obtained by Minkowski [14].

[8] This is not the limit of so-called Galilean electrodynamics, in which $c \to \infty$. The multiple possible meanings of Galilean electrodynamics are reviewed in [13].

to order v/c. These relations are not readily anticipated without use of Lorentz transformations.

In the present example there are no free charges or conduction currents associated with the slab in the lab frame, so we infer that $\mathbf{D} = 0$ and $\mathbf{H} = \mathbf{H}_0$ there. It follows from eq. (15) that the lab frame electric field inside the slab is,

$$\mathbf{E}_{\rm in} = -\left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\mathbf{v}}{c} \times \mathbf{H}_0, \tag{17}$$

and the voltage observed between the sliding contacts would be,

$$V = -E_{\rm in} d = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{vH_0 d}{c}\,, \tag{18}$$

where d is the thickness of the slab. This result differs from the naïve calculation (6) when the permeability μ_r is not unity.

2.1.3 Additional Comments on the Special-Relativistic Analysis

Inside the slab the magnetic field is $\mathbf{B}_{\rm in} = \mu_r \mathbf{H}_0 + \mathcal{O}(v^2/c^2)$, according to eq. (16). Outside the slab where $\epsilon_r = 1 = \mu_r$ we have $\mathbf{E}_{\rm out} = 0$ and $\mathbf{B}_{\rm out} = \mathbf{H}_0$, according to eqs. (15)-(16), as expected.

In the rest frame of the slab, the interior fields are, to order v/c,

$$\mathbf{D}^\star_{\rm in} = \frac{\mathbf{v}}{c} \times \mathbf{H}_0, \qquad \mathbf{E}^\star_{\rm in} = \frac{\mathbf{D}'_{\rm in}}{\epsilon} = \frac{\mathbf{v}}{\epsilon_r c} \times \mathbf{H}_0, \qquad \mathbf{H}^\star_{\rm in} = \mathbf{H}_0. \qquad \mathbf{B}^\star_{\rm in} = \mu_r \mathbf{H}_{\rm in} = \mu_r \mathbf{H}_0. \tag{19}$$

using eqs. (10)-(11), while the exterior fields are,

$$\mathbf{D}^\star_{\rm out} = \mathbf{E}^\star_{\rm out} = \frac{\mathbf{v}}{c} \times \mathbf{H}_0 = \mathbf{D}^\star_{\rm in}, \qquad \mathbf{B}^\star_{\rm out} = \mathbf{H}^\star_{\rm out} = \mathbf{H}_0 = \mathbf{H}^\star_{\rm in}. \tag{20}$$

The presence of a nonzero (and uniform) electric displacement $\mathbf{D}^\star$ in the rest frame of the slab is possibly surprising, as we argued that there is no free charge associated with the slab. However, there must be conduction current density $\mathbf{J}$ somewhere in the lab frame to generate the nominally uniform magnetic field $\mathbf{H}_0$. Recalling that charge density ρ and current density $\mathbf{J}$ are combined in a 4-vector $(\rho, \mathbf{J}/c)$, there is an apparent free charge density ρ' in the rest frame of the slab given by,

$$\rho^\star = \gamma\left(\rho - \frac{\mathbf{v} \cdot \mathbf{J}}{c^2}\right), \tag{21}$$

which leads to a nonzero field $\mathbf{D}^\star$. It might appear that the part of the apparent charge density $\rho^\star$ in the rest frame associated with current $\mathbf{J}$ in the lab frame is negligible in the low-velocity limit. However, the following idealized example shows that this is not so.

We imagine that the uniform magnetic field $\mathbf{H}_0$ is generated in the lab frame by two large, conducting sheets that carry current $\mathbf{J} = \pm cH_0\,\hat{\mathbf{v}}/4\pi d$, where d is the thickness of the sheets, which are on either side of the moving slab. If the charge density ρ is zero on these current sheets, then the apparent charge density[9] in the rest frame of the slab is $\rho^\star \approx \mp H_0 v/4\pi c d$, and the corresponding surface charge density is $\sigma^\star \approx \mp H_0 v/4\pi c$. Thus, the sheets act like a parallel-plate capacitor in the rest frame, with electric displacement $\mathbf{D}^\star = 4\pi\sigma^\star = \mathbf{v}/c \times \mathbf{H}_0$ between the sheets, and hence surrounding the slab.

We can also consider the densities $\mathbf{P} = (\mathbf{D} - \mathbf{E})/4\pi$ and $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$ of electric and magnetic polarization in the slab. These densities can also be expressed in terms of the permittivity ϵ_r and permeability μ_r using the constitutive equations (7)-(8) in the comoving inertial frame of the slab, and (15)-(16) in the lab frame. Thus,

$$\mathbf{P}^\star_{\rm in} = \frac{\epsilon_r - 1}{4\pi}\mathbf{E}^\star_{\rm in} = \frac{\epsilon_r - 1}{4\pi\epsilon_r}\frac{\mathbf{v}}{c} \times \mathbf{H}_0, \qquad \mathbf{M}^\star_{\rm in} = \frac{\mu_r - 1}{4\pi}\mathbf{H}'_{\rm in} = \frac{\mu_r - 1}{4\pi}\mathbf{H}_0, \tag{22}$$

and,

$$\mathbf{P}_{\rm in} = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\mathbf{v}}{4\pi c} \times \mathbf{H}_0 = -\frac{\mathbf{E}_{\rm in}}{4\pi}, \qquad \mathbf{M}_{\rm in} = \frac{\mu_r - 1}{4\pi}\mathbf{H}_{\rm in} + (\epsilon_r\mu_r - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}_{\rm in} \approx \frac{\mu_r - 1}{4\pi}\mathbf{H}_0. \tag{23}$$

Note that the electric polarization in the lab frame does not obey the form $\mathbf{P} = (\epsilon_r - 1)\mathbf{E}/4\pi$, as might have been supposed to hold if relativistic effects were ignored. Thus, the analysis of sec. 2.1.1 was indeed too naïve.

Note that the lab-frame electric polarization is nonzero when $\epsilon_r = 1$, so long as μ_r differs from unity. This is an example of the well-known phenomenon that a moving magnetization is associated with an electric polarization (see, for example, sec. 88 of [7] or sec. 18.6 of [11]).

The polarizations (22)-(23) in the two frames are also consistent with the Lorentz transformation of the polarization tensor [7],

$$\mathbf{P}^\star_\parallel = \mathbf{P}_\parallel, \quad \mathbf{P}^\star_\perp = \gamma\left(\mathbf{P}_\perp - \frac{\mathbf{v}}{c} \times \mathbf{M}\right), \quad \mathbf{M}^\star_\parallel = \mathbf{M}_\parallel, \quad \mathbf{M}^\star_\perp = \gamma\left(\mathbf{M}_\perp + \frac{\mathbf{v}}{c} \times \mathbf{P}\right). \tag{24}$$

2.1.4 Why is There a Nonzero Electric Field in the Lab Frame?

The lab-frame electric field (17) inside the moving slab is nonzero for both the cases of $\epsilon_r \neq 1$, $\mu_r = 1$ and $\epsilon_r = 1$, $\mu_r \neq 1$.

When the medium can be polarized electrically ($\epsilon_r \neq 1$), we can invoke the sense of the naïve argument of sec. 2.1.1. The Lorentz force on the moving atoms deforms those atoms, creating a nonzero electric polarization $\mathbf{P}_{\rm in}$ in the lab frame, and hence a nonzero electric field $\mathbf{E}_{\rm in}$.

If, however, the medium cannot be polarized electrically ($\epsilon_r = 1$), but can be polarized magnetically ($\mu_r \neq 1$), the argument is more subtle. The Ampèreian view is that magnetic polarization is associated with "molecular currents" $\mathbf{J}^\star_{\rm mol}$ (in the rest frame of the medium).

[9]The total charge associated with the sheets is zero both in the lab frame and in the rest frame. A more realistic configuration of the currents is that the two conducting sheets are joined at their edges that are perpendicular to $\mathbf{v}$ by two other conducting sheets, also carrying current J, so as to form a complete circuit. Then, if the slab is accelerated from rest in the lab frame up to velocity $\mathbf{v}$, an observer on the slab would consider that some charge migrated from one of the conducting sheets parallel to $\mathbf{v}$ to the other, while conserving total charge.

An electrically neutral, but magnetically polarized medium that is in motion appears, according to the inverse of eq. (21), $\rho = \gamma(\rho^\star + \mathbf{v} \cdot \mathbf{J}^\star/c^2) = \gamma \mathbf{v} \cdot \mathbf{J}^\star/c^2$, to have regions of nonzero charge density in the lab frame, which leads to an electric field there.

This discussion reinforces the conclusion that the naïve argument (5)-(6) is incorrect because it predicts no electric field/voltage in the lab frame when $\epsilon_r = 1$ but $\mu_r \neq 1$.

While the nonzero lab-frame electric field in the moving slab can be called a "relativistic" effect in both cases $\epsilon_r \neq 1$, $\mu_r = 1$ and $\epsilon_r = 1$, $\mu_r \neq 1$, an understanding of the latter case emerged only after the development of the theory of special relativity.

2.2 Rotating Cylindrical Shell

The expression (18) has not been tested directly in the laboratory. However, the experiment of Wilson and Wilson [4] has confirmed that the voltage observed between contacts sliding on a cylindrical shell of inner radius a and outer radius b that rotates with angular velocity ω in an external magnetic field $\mathbf{H}_0$ parallel to its axis is,

$$V = -\int_a^b E_r(r)\, dr = \left(\mu_r - \frac{1}{\epsilon_r}\right) \frac{\omega H_0 (b^2 - a^2)}{2c}\,. \tag{25}$$

2.2.1 Analysis Using a Local Inertial Frame

The experimental result (25) follows immediately from eq. (17) on writing $\mathbf{v} = \omega r\, \boldsymbol{\phi}$,

$$\mathbf{E}_{\rm in} = -\left(\mu_r - \frac{1}{\epsilon_r}\right) \frac{\omega r\, \boldsymbol{\phi}}{c} \times \mathbf{H}_0 = -\left(\mu_r - \frac{1}{\epsilon_r}\right) \frac{\omega r\, H_0}{c}\, \hat{\mathbf{r}}. \tag{26}$$

Hence, the Wilson-Wilson experiment was historically considered as evidence for the correctness of (non)relativistic electrodynamics of moving media.

However, it is possible to be surprised at the success of an analysis based on special relativity when a noninertial (rotating) frame is involved. Perhaps an analysis based on general covariance should be used in this case.[10]

A valid technique of general covariance when describing a rotating system is to associate each line with fixed (r and ϕ) in the rotating system with a nonrotating, inertial frame whose (vector) velocity relative to the lab frame is that of the line (relative to the lab frame). This description in terms of what I call local inertial frames is particularly useful when the physical quantities of interest at a point in the rotating system are determined largely by the (local) behavior of the system close to that point.

In the experiment of Wilson and Wilson, the voltage difference in the lab frame is related to the radial electric field E_r in the lab frame along the line between the sliding contacts. The lab-frame electric fields $\mathbf{D}$ and $\mathbf{E}$ at a point along the line of integration of $V = \int_a^b E_r\, dr$ can be related to the electric fields $\mathbf{D}^\star$ and $\mathbf{E}^\star$ in the local inertial frame by a Lorentz transformation. The only physics needed in the local inertial frame of a point P along the line of integration is the constitutive equation (7), $\mathbf{D}^\star = \epsilon_r \mathbf{E}^\star$, which is based on a

[10]In physics, general covariance is usually associated with Einstein's theory of general relativity. If the latter is taken to be the theory of gravity due to mass, then the present example is not one of general relativity (and the centrifugal force experienced in the rotating frame is not a kind of gravitational force).

macroscopic average over a volume containing only a few atoms, assumed to be at rest. Atoms at a large distance from point P are not at rest in the local inertial frame, and the constitutive equation (7) is not valid for those atoms in this frame. However, this awkward fact is irrelevant to our use of eq. (7) at point P. Hence, we expect that an analysis based on the use of local inertial frames will give a valid result when transformed to the lab frame, as is confirmed by experiment.

Another well-known example of successful use of local inertial frames to describe a relativistic rotating system is a muon storage ring. Here, a beam of mu mesons moves in a circle at speeds very close to that of light under the influence of a constant laboratory magnetic field. Mu mesons decay with a lifetime $\tau_\mu = 2.2\,\mu$s *when at rest, while mu mesons circulating with velocity* v *are observed in the lab frame to decay with a lifetime* $\tau = \gamma\tau_\mu$ *where* $\gamma = 1/\sqrt{1 - v^2/c^2}$. *This is the result expected from the special-relativistic time dilation in the local inertial frame of a circulating mu meson. The physics of the decay of a mu meson is governed by extremely short distance scales, so it is no surprise that the use of local inertial frames is effective in this case.*

Other reviews of the use of local inertial frames to study the electrodynamics of rotating systems are [12, 15]. For other discussions particular to the Wilson-Wilson experiment, see [16]-[19].

Bound and Free Charges *(The rest of sec. 2.2.1 was rewritten Nov. 9, 2020, as inspired by e-discussions with Jacques Maritz.)*

The electric polarization in the lab frame of the rotating cylinder is again given by eq. (23). There is also a nonzero bound charge density in the lab frame,

$$\rho_{\rm bound} = -\boldsymbol{\nabla}\cdot\mathbf{P}_{\rm in} = -\frac{1}{4\pi}\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm in} = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega H_0}{2\pi c}, \qquad (27)$$

recalling eq. (26). The cylindrical surfaces at $r = a$ and p support bound surface charge densities,

$$\sigma_{\rm bound}(r = a^+) = -\mathbf{P}_{\rm in}(r = a)\cdot\hat{\mathbf{r}} = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega a H_0}{4\pi c}, \qquad (28)$$

$$\sigma_{\rm bound}(r = b^-) = \mathbf{P}_{\rm in}(r = b)\cdot\hat{\mathbf{r}} = -\left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega b H_0}{4\pi c}. \qquad (29)$$

The bound surface charges are,

$$Q_{\rm bound}(r = a^+) = 2\pi a l \sigma_{\rm bound}(r = a^+) = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega a^2 l H_0}{2}, \qquad (30)$$

$$Q_{\rm bound}(r = b^-) = 2\pi b l \sigma_{\rm bound}(r = b) = -\left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega b^2 l H_0}{2c}. \qquad (31)$$

The total bound charge is zero, as expected,

$$\begin{aligned} Q_{\rm bound} &= \int \rho_{\rm bound}\, d{\rm Vol} + Q_{\rm bound}(r = a^+) + Q_{\rm bound}(r = b^-) \\ &= \int_0^b \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega H_0}{2\pi c}\, 2\pi l r\, dr + \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega(a^2 - b^2) l H_0}{2c} = 0. \qquad (32) \end{aligned}$$

In the Wilson-Wilson experiment [4], the surfaces at $r = a$ and b of the rotating cylinder were metalized, forming a cylindrical capacitor of capacitance,

$$C \approx \frac{\epsilon_r l}{2\ln(b/a)}, \tag{33}$$

independent of the angular velocity ω. In the rest of this section, such conductors are assumed to exist. The bound surface charge (30)-(31) induces free charge on these conductors, given by,

$$Q_{\rm free}(r = a^-) = -Q_{\rm bound}(r = a^+) = -\left(\mu_r - \frac{1}{\epsilon}\right)\frac{\omega a^2 l H_0}{2}, \tag{34}$$

$$Q_{\rm free}(r = b^+) = -Q_{\rm bound}(r = b^-) = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega b^2 l H_0}{2c}. \tag{35}$$

The total free charge on the rotating capacitor is nonzero,

$$Q_{\rm free} = Q_{\rm free}(r = a^-) + Q_{\rm free}(r = b^+) = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega(b^2 - a^2) l H_0}{2}, \tag{36}$$

unlike an ordinary capacitor which has equal and opposite free charges $\pm Q$ on its two electrodes.

Circuit Analysis

If the rotating capacitor is part of an "ungrounded" electrical circuit with zero total free charge, the free charge (36) on the conductors of the rotating cylinder implies that equal and opposite free charge resides elsewhere on the circuit, which compensating charge can be thought of as associated with the stray capacitance of the circuit to "ground".

The rotating cylinder acts as a (unipolar) voltage source of strength (25), which is in series with its capacitance. The energy which may be provided by this source comes from the mechanism that drives the constant angular velocity ω, or, if no such mechanism exists, from the (decreasing) mechanical kinetic energy of the rotating cylinder as ω falls to zero.

If there is no other voltage source in the circuit, there is no charge $\pm Q$ on the conductors of the rotating cylinder. But when another voltage source is present, the charge on the outer conductor of the rotating cylinder is $Q_{\rm free}(r = b^+)$ of eq. (35) plus an additional (free) charge Q, while that on the inner conductor is $Q_{\rm free}(r = a^-)$ of eq. (34) plus an additional (free) charge $-Q$, and the voltage drop across the rotating cylinder is V of eq. (25) plus Q/C.

Variable Rotation

The angular velocity ω was constant in the Wilson-Wilson experiment, but if it were time dependent, then the rotating cylinder would be a kind of variable voltage source (in series with the constant capacitance C of eq. (33)).[11]

[11] An oddity in case of variable velocity ω is that the current into the outer conductor of the rotating cylinder must include the term $dQ_{\rm free}(r = b^+)/dt = (\mu_r - 1/\epsilon_r)\,\dot\omega b^2 l H_0/2$, and the current out of the inner conductor must include $dQ_{\rm free}(r = a^-)/dt = (\mu_r - 1/\epsilon_r)\,\dot\omega a^2 l H_0/2$.

2.2.2 Analysis in the Rotating Frame

A different analysis of the Wilson-Wilson experiment could involve a transformation between the lab and the rotating frame, along with knowledge of the constitutive equation in the rotating frame.

One of the first analyses of electrodynamics in a rotating frame using general covariance was made by Schiff [20]. See also [21]-[38]. A major concern of these efforts is the form of Maxwell's equations in a rotating frame, which will not be needed here as we do not deduce the fields in the rotating frame from sources in that frame. Those sources include "fictitious" terms needed to explain, for example, how a magnetic field exists in the rest frame of a rotating sphere of charge.

We do need the transformation of the fields **B**, **D**, **E** and **H** between the lab frame and the rotating frame. We record here only the low-velocity ($\mathcal{O}(v/c)$) limit of these transformations [36, 37, 39]:

$$\mathbf{B} = \mathbf{B}', \qquad \mathbf{D} = \mathbf{D}' - \frac{\mathbf{v}}{c} \times \mathbf{H}', \qquad \mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}', \qquad \mathbf{H} = \mathbf{H}', \tag{37}$$

where a $'$ indicates quantities observed in the rotating frame, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ is the velocity of the point of observation with respect to the lab frame.

We also need the constitutive equations in the rotating frame. We might suppose that these have the forms (7)-(8) because the rotating frame is a rest frame of the medium. However, we recall that "fictitious" forces appear to observers in rotating frames, so the constitutive equations (which relate fields/forces to dipole source terms) in a rotating frame may not have the same form when the medium is at rest in an inertial frame. As discussed in sec. A.5 of [39], the constitutive equations in a rotating frame can be written to $\mathcal{O}(v/c)$ as,

$$\mathbf{D}' = \epsilon_r \mathbf{E}', \tag{38}$$

$$\mathbf{B}' = \mu_r \mathbf{H}' - (\epsilon_r \mu_r - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}', \tag{39}$$

Then, transformation of the fields from the rotating frame to the lab frame leads to the forms (15)-(16) obtained by the methods of special relativity.[12] Hence, the predictions for the lab-frame voltage drop between the sliding contacts of the Wilson-Wilson experiment are the same for analyses based on use of local inertial frames and on use of the rotating frame.

2.2.3 Additional Comments on the Analysis in the Rotating Frame

The fields $\mathbf{B}'$, $\mathbf{D}'$, $\mathbf{E}'$ and $\mathbf{H}'$ in the rotating frame are, to order v/c, the same as the fields (19)-(20) in a local inertial frame.

Using the relations $\mathbf{D}' = \mathbf{E}' + 4\pi \mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi \mathbf{M}'$ for the electric and magnetic polarization densities, the constitutive equations (38)-(39 imply that to order v/c,

$$\mathbf{P}' = \frac{\epsilon_r - 1}{4\pi}\mathbf{E}', \qquad \mathbf{M}' = \frac{\mu_r - 1}{4\pi}\mathbf{H}' + \frac{\epsilon_r \mu_r - 1}{4\pi}\frac{\mathbf{v}}{c} \times \mathbf{E}'. \tag{40}$$

[12] When this analysis is done to all orders of v/c, the forms (13)-(14) are obtained [36].

Because $E' = vH_0/\epsilon_r c$ in the Wilson-Wilson experiment, the magnetization M' observed in the rotating frame is the same $M^\star$ observed in the local inertial frame to order v/c. However, in examples with nonzero free charge (and $\epsilon_r\mu_r$ different from unity), the magnetization would appear to be different in the rotating frame and the local inertial frame.

If we accept that the magnetic field is the same in the rotating frame as in the lab frame, then we can use Maxwell's equations in the rotating frame to solve for the electric fields $\mathbf{D}'$ and $\mathbf{E}'$ and the polarization $\mathbf{P}'$. In particular,

$$\boldsymbol{\nabla}' \cdot \mathbf{D}' = 4\pi \left(\rho'_{\rm free} + \frac{\boldsymbol{\omega} \cdot \mathbf{H}'}{2\pi c} + \frac{\mathbf{v}}{4\pi c} \cdot \boldsymbol{\nabla}' \times \mathbf{H}' \right) = \frac{2\boldsymbol{\omega} \cdot \mathbf{H}'}{c} = \frac{2\omega H_0}{c}\,, \tag{41}$$

expresses that the electric field has "fictitious" source terms in the rotating frame [20, 39]. The symmetry of the problem implies that $\mathbf{D}'$ is purely radial, so that eq. (41) leads to,

$$\mathbf{D}' = \frac{\omega r' H_0}{c}\,\hat{\mathbf{r}}' = \frac{\mathbf{v}}{c} \times \mathbf{H}_0. \tag{42}$$

This result is consistent with the (inverse of the) transformation (37) that $\mathbf{D}' = \mathbf{D} + \mathbf{v}/c \times \mathbf{H}_0$, recalling that in the lab frame $\mathbf{D} = 0$ since there are no free charges. We then have that,

$$\mathbf{E}' = \frac{\mathbf{D}'}{\epsilon_r} = \frac{1}{\epsilon_r}\frac{\mathbf{v}}{c} \times \mathbf{H}_0, \qquad \text{and} \qquad \mathbf{P}' = \frac{(\epsilon_r - 1)\mathbf{D}'}{4\pi\epsilon_r} = \frac{(\epsilon_r - 1)}{4\pi\epsilon_r}\frac{\mathbf{v}}{c} \times \mathbf{H}_0. \tag{43}$$

Accepting that $\mathbf{H}' = \mathbf{H}_0$, we have that $\mathbf{B}' = \mathbf{B}_0 = \mu_r \mathbf{H}_0$ and $\mathbf{M}' = \mathbf{M}_0 = (\mu_r - 1)\mathbf{H}_0/4\pi$. Transforming the electric field $\mathbf{E}'$ and polarization $\mathbf{P}'$ to the lab frame [39], we find,

$$\mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}' = -\left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\mathbf{v}}{c} \times \mathbf{H}_0, \quad \text{and} \quad \mathbf{P} = \mathbf{P}' + \frac{\mathbf{v}}{c} \times \mathbf{M}' = \left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\mathbf{v}}{4\pi c} \times \mathbf{H}_0, \tag{44}$$

as found in eqs. (17) and (23).

There is also a nonzero bound charge density in the rotating frame,

$$\rho'_{\rm bound} = -\boldsymbol{\nabla}' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{M}' = -\left(\mu_r - \frac{1}{\epsilon_r}\right)\frac{\omega r H_0}{4\pi c} = \rho_{\rm bound}\,, \tag{45}$$

using eq. (46) of [39], which agrees with the lab-frame charge density (27).

The important overall conclusion is that the same understanding of lab-frame electrodynamics of a rotating system can be gained either by use of local inertial frames or by use of the rotating frame, but since the use of local inertial frames is conceptually simpler this approach is to be preferred in practice.

References

[1] W. Weber, *Unipolare Induction*, in *Resultate aus der Beobachtungen des magnetishes Vereins im Jahre 1839*, C.F. Gauss and W. Weber, eds. (Weidemanns, Leipzig, 1940), p. 63, http://kirkmcd.princeton.edu/examples/EM/weber_40.pdf
Ann. d. Phys. **52**, 353 (1841), http://kirkmcd.princeton.edu/examples/EM/weber_ap_52_353_41.pdf

[2] M. Faraday, *On Lines of Magnetic Force*, Phil. Trans. Roy. Soc. London **142**, 25 (1852), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_142_25_52.pdf

[3] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux-Rule Exception?* (July 27, 2019), http://kirkmcd.princeton.edu/examples/faradaydisk.pdf

[4] M. Wilson and H.A. Wilson, *On the Electric Effect of Rotating a Magnetic Insulator in a Magnetic Field*, Proc. Roy. Soc. London A **89**, 99 (1913), http://kirkmcd.princeton.edu/examples/EM/wilson_prsla_89_99_13.pdf

[5] H.A. Wilson, *On the Electric Effect of Rotating a Dielectric in a Magnetic Field*, Proc. Roy. Soc. London **73**, 490 (1904), http://kirkmcd.princeton.edu/examples/EM/wilson_prsl_73_490_04.pdf

[6] J.B. Hertzberg *et al.*, *Measurement of the relativistic potential difference across a rotating magnetic dielectric cylinder*, Am. J. Phys. **69**, 648 (2001), http://kirkmcd.princeton.edu/examples/EM/hertzberg_ajp_69_648_01.pdf

[7] R. Becker, *Electromagnetic Fields and Interactions* (Dover, 1964).

[8] D.R. Corson, *Electromagnetic Induction in Moving Systems*, Am. J. Phys. **24**, 126 (1956), http://kirkmcd.princeton.edu/examples/EM/corson_ajp_24_126_56.pdf

[9] E.M. Pugh, *Electromagnetic Relations in a Single Coordinate System*, Am. J. Phys. **32**, 879 (1964), http://kirkmcd.princeton.edu/examples/EM/pugh_ajp_32_879_64.pdf

[10] M.J. Crooks *et al.*, *One-piece Faraday generator: A paradoxical experiment from 1851*, Am. J. Phys. **46**, 729 (1978), http://kirkmcd.princeton.edu/examples/EM/crooks_ajp_46_729_78.pdf

[11] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2$^{\rm nd}$ ed. (Addison-Wesley, 1962), kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[12] J. van Bladel, *Relativistic Theory of Rotating Disks*, Proc. IEEE **61**, 260 (1973), http://kirkmcd.princeton.edu/examples/EM/vanbladel_pieee_61_260_73.pdf

[13] M. Le Bellac and J.-M. Lévy-Leblond, *Galilean Electrodynamics*, Nuovo Cim. **14B**, 217 (1973), http://kirkmcd.princeton.edu/examples/EM/lebellac_nc_14b_217_73.pdf

[14] H. Minkowski, *Die Grundgleichen für die elektromagnetischen Vorlage in bewegten Körpern*, Göttinger Nachricthen, pp. 55-116 (1908), http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08.pdf http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08_english.pdf

[15] T. Shiozawa, *Phenomenological and Electron-Theoretical Study of the Electrodynamics of Rotating Systems*, Proc. IEEE **61**, 1694 (1973), http://kirkmcd.princeton.edu/examples/EM/shiozawa_pieee_61_1694_73.pdf

[16] T.A. Weber, *Measurements on a rotating frame in relativity, and the Wilson and Wilson experiment*, Am. J. Phys. **65**, 946 (1997), http://kirkmcd.princeton.edu/examples/EM/weber_ajp_65_946_97.pdf

[17] P. Hillion, *The Wilsons' Experiment*, Apeiron **6**, 1 (1999),
http://kirkmcd.princeton.edu/examples/EM/hillion_apeiron_6_1_99.pdf

[18] C.E.S. Canovan and R.W. Tucker, *Maxwell's equations in a uniformly rotating dielectric medium and the WilsonWilson experiment*, Am. J. Phys. **78**, 1181 (2010),
http://kirkmcd.princeton.edu/examples/EM/canovan_ajp_78_1181_10.pdf

[19] H. Heumann and S. Kurz, *Modeling and Finite-Element Simulation of the WilsonWilson Experiment*, IEEE Trans. Mag. **50**, 7001404 (2014),
http://kirkmcd.princeton.edu/examples/EM/heumann_ieeetm_50_7001404_14.pdf

[20] L.I. Schiff, *A Question in General Relativity*, Proc. Nat. Acad. Sci. **25**, 391 (1939),
http://kirkmcd.princeton.edu/examples/EM/schiff_proc_nat_acad_sci_25_391_39.pdf

[21] D. van Dantzig and R. Koninkl, *Electromagnetism Independent of Metrical Geometry*, K. Akad. Amsterdam Proc. **37**, 521, 643 (1934),
http://kirkmcd.princeton.edu/examples/EM/vandantzig_kaap_37_521_34.pdf

[22] M.G. Trocheris, *Electrodynamics in a Rotating Frame of Reference*, Phil. Mag. **40**, 1143 (1949), http://kirkmcd.princeton.edu/examples/EM/trocheris_pm_40_1143_49.pdf

[23] J. Ise and J.L. Uretsky, *Vacuum Electrodynamics on a Merry-Go-Round*, Am. J. Phys. **26**, 431 (1958), http://kirkmcd.princeton.edu/examples/EM/ise_ajp_26_431_58.pdf

[24] D.L. Webster, *Schiff's Charges and Currents in Rotating Matter*, Am. J. Phys. **31**, 590 (1963), http://kirkmcd.princeton.edu/examples/EM/webster_ajp_31_590_63.pdf

[25] W.H. Irvine, *Electrodynamics in a Rotating Frame of Reference*, Physica **30**, 1160 (1964), http://kirkmcd.princeton.edu/examples/EM/irvine_physica_30_1160_64.pdf

[26] T.C.. Mo, *Theory of Electrodynamics in Media in Noninertial Frames and Applications*, J. Math. Phys. **11**, 25889 (1970), http://kirkmcd.princeton.edu/examples/EM/mo_jmp_11_25889_70.pdf

[27] G.E. Modesitt, *Maxwell's Equations in a Rotating Frame*, Am. J. Phys. **38**, 1487 (1970),
http://kirkmcd.princeton.edu/examples/EM/modesitt_ajp_38_1487_70.pdf

[28] D.L. Webster and R.C. Whitten, *Which Electromagnetic Equations Apply in Rotating Coordinates?*, Astro. Space Sci. **24**, 323 (1973),
http://kirkmcd.princeton.edu/examples/EM/webster_ass_24_323_73.pdf

[29] H.A. Atwater, *The Electromagnetic Field in Rotating Coordinate Frames*, Proc. IEEE **63**, 316 (1975), http://kirkmcd.princeton.edu/examples/EM/atwater_pieee_63_316_75.pdf

[30] Ø. Grøn, *Application of Schiff's Rotating Frame Electrodynamics*, Int. J. Theor. Phys. **23**, 441 (1984), http://kirkmcd.princeton.edu/examples/EM/gron_ijtp_23_441_84.pdf

[31] A. Georgiou, *The Electromagnetic Field in Rotating Coordinates*, Proc. IEEE **76**, 1051 (1988), http://kirkmcd.princeton.edu/examples/EM/georgiou_pieee_76_1051_88.pdf

[32] G.C. Scorgie, *Electromagnetism in non-inertial coordinates*, J. Phys. A **23**, 5169 (1990), http://kirkmcd.princeton.edu/examples/EM/scorgie_jpa_23_5169_90.pdf

[33] H.W. Crater, *General covariance, Lorentz covariance, the Lorentz force, and the Maxwell equations*, Am. J. Phys. **62**, 923 (1994), http://kirkmcd.princeton.edu/examples/EM/crater_ajp_62_923_94.pdf

[34] G.N. Pellegrini and A.R. Swift, *Maxwell's equations in a rotating medium: Is there a problem?*, Am. J. Phys. **63**, 694 (1995), http://kirkmcd.princeton.edu/examples/EM/pellegrini_ajp_63_694_95.pdf

[35] M.L. Burrows, *Comment on "Maxwell's equations in a rotating medium: Is there a problem?"*, Am. J. Phys. **65**, 929 (1997), http://kirkmcd.princeton.edu/examples/EM/burrows_ajp_65_929_97.pdf

[36] C.T. Ridgely, *Applying relativistic electrodynamics to a rotating material medium*, Am. J. Phys. **66**, 114 (1998), http://kirkmcd.princeton.edu/examples/EM/ridgely_ajp_66_114_98.pdf

[37] C.T. Ridgely, *Applying covariant versus contravariant electromagnetic tensors to rotating media*, Am. J. Phys. **67**, 414 (1999), http://kirkmcd.princeton.edu/examples/EM/ridgely_ajp_67_414_99.pdf

[38] D.V. Redžić, *Electromagnetism of rotating conductors revisited*, Eur. J. Phys. **23**, 127 (2002), http://kirkmcd.princeton.edu/examples/EM/redzic_ejp_23_127_02.pdf

[39] K.T. McDonald, *Electrodynamics of Rotating Systems* (Aug. 6, 2008), http://kirkmcd.princeton.edu/examples/rotatingEM.pdf