Charge Density in a Current-Carrying Wire

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1 Problem

Discuss the volume densities ρ_+ and $\rho_- < 0$ of positive and negative electric charges in a wire that carries a steady current, assuming that in the lab frame the positive charges are at rest and the current is due to negative charges (electrons) that all have speed v.

This problem seems to have been first considered in 1877 by Clausius [1, 2], who argued that the force on an electric charge at rest outside the wire is zero in the lab frame of the wire, but it was noticed [3, 4] that this implies a nonzero force in the rest frame of the charges whose motion in the lab constitutes the electrical current there.[1]

2 Solution

Discussions of the force on a charged particle outside a current-carrying wire often assume that the wire is electrically neutral.[2] This problem explores how this assumption is not quite correct for resistive, current-carrying wires.

We give solutions both in the lab frame and in the rest frame of the conduction electrons, using both Maxwell's equations and special relativity.

We note that for current to flow in a resistive wire, there must be an axial electric field inside the wire, which requires a surface charge distribution that varies with position along the wire. See, for example, [8]-[49]. The surface charge distribution could include a uniform term of any magnitude. These surface charges are kept from leaving the surface by quantum effects often summarized by the term "work function." Likewise, the positive charges in the interior of the wire are held together in a lattice by quantum effects. We suppose that the positive charge density ρ_+ is uniform in the lab frame.

In this problem we assume that the conduction electrons can be described classically. Then, for steady axial motion, there must be zero radial force on these electrons.

We use a cylindrical coordinate system (r, ϕ, z) whose axis is the axis of the wire. We suppose that the flow of conducting electrons is purely axial, and azimuthally symmetric. The negative charge density, ρ_-, could depend on the radius r. The electric field **E** has no azimuthal component, while the magnetic field **B** has only an azimuthal component.

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[1]A related issue is that a conductor moving in a (static) magnetic field is also not quite electrically neutral [5, 6, 7].

[2]This assumption was advocated by Clausius [1], who also argued that electrical current is due only to the flow of one sign of charge, as can be assumed in this problem. See the Appendix for discussion of currents of both signs.

2.1 Lab Frame

For steady flow of current, the axial component of the electric field must be independent of z. Then, assuming that there is no azimuthal component to the electric field, Maxwell's first equation tells us that,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = 4\pi\rho(r) = 4\pi[\rho_+ + \rho_-(r)] \tag{1}$$

(in Gaussian units). The radial component of the electric field vanishes at $r = 0$, so we find that,

$$E_r(r) = \frac{4\pi}{r}\int_0^r r'[\rho_+ + \rho_-(r')]\,dr' \tag{2}$$

(in Gaussian units).

The azimuthal magnetic field is due to the motion of the negative charges with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, and Ampère's law implies that,

$$B_\phi(r) = \frac{4\pi v}{cr}\int_0^r r'\rho_-(r')\,dr'. \tag{3}$$

The Lorentz force density on the negative charges is,

$$\begin{aligned}
\mathbf{f}_-(r) &= \rho_-(r)\left(\mathbf{E}(r) + \frac{\mathbf{v}}{c}\times\mathbf{B}(r)\right) \\
&= \rho_-(r)\left\{\frac{4\pi\,\hat{\mathbf{r}}}{r}\int_0^r r'\left[\rho_+ + \rho_-(r')\left(1 - \frac{v^2}{c^2}\right)\right]dr' + E_z(r)\,\hat{\mathbf{z}}\right\}.
\end{aligned} \tag{4}$$

For steady axial motion of the conduction electrons, the radial force on them must vanish at all r inside the wire, which implies that the negative charge density is uniform with value,

$$\rho_- = -\frac{\rho_+}{1 - v^2/c^2} = -\gamma^2\rho_+, \tag{5}$$

and the total charge density is,[3]

$$\rho = \rho_+ + \rho_- = \rho_-\frac{v^2}{c^2}, \tag{6}$$

where,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{7}$$

The positive charge density is less than the negative by one part in 10^{21} for $v = 1$ cm/s. This corresponds roughly to 10 more electrons than protons in each cubic millimeter of copper wire.[4]

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[3]This result may have first appeared in [17].

[4]Some authors appear to argue that if a system of battery and wire is electrically neutral before the battery is connected (unlikely to be true in practice), with the initial charge densities ρ_+ and ρ_- being "exactly" equal, it is impossible that this condition changes, even by one part in 10^{21}, after the battery is connected. Some of this debate is reviewed in [47].

Using eq. (5) in eqs. (2)-(3), the radial electric field and azimuthal magnetic field inside the wire are,

$$E_r(r) = -2\pi(\gamma^2 - 1)\rho_+ r, \qquad B_\phi(r) = -\frac{2\pi\gamma^2 v \rho_+ r}{c}. \tag{8}$$

Because of the small difference between the positive and negative charge densities, the positive charges experience a small inward radial force density (in addition to the axial force due to collisions with the conduction electrons),

$$f_{+,r} = \rho_+ E_r(r) = -2\pi(\gamma^2 - 1)\rho_+^2 r = -\frac{2\pi\gamma^2 v^2 \rho_+^2 r}{c^2}. \tag{9}$$

This example tacitly assumes that the wire can support an internal magnetic field, which requires the wire to have finite conductivity σ. In this case, there must exist a longitudinal electric field inside the wire of magnitude $E_z = I/\pi a^2 \sigma$, where a is the radius of the wire. This longitudinal electric field could be due to a "battery", or it could be induced by a changing magnetic field $B_{ext}(t)$ external to the wire. In the latter case, there would be a (time-dependent) $\mathbf{I} \times \mathbf{B}_{ext}$ force transverse to the wire axis and a (time-dependent) internal electric field transverse to the axis to cancel this force, as noted by Hall [50] for steady external fields. We do not pursue this case further here, but content ourselves by supposing the current is due to a "battery" that can supply whatever charge is needed to maintain the longitudinal electric field E_z, which is associated with a surface electric charge density [35], such that the linear charge density $\lambda(z)$ of the wire need not be zero.

The results of this section were obtained without use of special relativity (although we use the symbol γ, eq. (7), commonly associated with that theory), and they are based on only the equations of electro- and magnetostatics. However, they do assume that electrical currents can be regarded as due to moving electrical charges (convection currents), which was not considered to be demonstrated until the work of Rowland in 1876 [51, 52]. All of the premises of this section also hold in so-called Galilean electrodynamics, formulated in 1973 [53], and hence the conclusions of this section also hold in that view.

2.2 Rest Frame of the Conduction Electrons

We denote quantities in the rest frame of the conduction electrons with a *. Thus, the velocity of the positively charged particles in the * frame is $\mathbf{v}_+^\star = -v\,\hat{\mathbf{z}}$.

If the radial electric field $E_r^\star$ were nonzero the conduction electrons would experience a radial force. Hence, we expect the radial electric field, and the bulk charge density $\rho^\star$, to vanish in the * frame.

The Lorentz contraction of a moving "stick" results in an observer of a "stick of charge" reporting a charge density larger by a factor γ than that in the rest frame of the "stick." Thus, the density ρ_- of negative charge in the lab frame is larger than that in the * frame,

$$\rho_- = \gamma\rho_-^\star \left[= \gamma\left(\rho_-^\star + \frac{\mathbf{v} \cdot \mathbf{J}_-^\star}{c^2}\right)\right], \tag{10}$$

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since $\mathbf{J}_-^\star = 0$ in the $\star$ frame. In contrast, the density $\rho_+^\star$ of positive charge in the $\star$ frame is larger than that in the lab frame,

$$\rho_+^\star = \gamma \rho_+ \left[= \gamma \left(\rho_+ - \frac{\mathbf{v} \cdot \mathbf{J}_+}{c^2} \right) \right], \tag{11}$$

since $\mathbf{J}_+ = 0$ in the lab frame. The total charge density in the $\star$ frame is,

$$\rho^\star = \rho_+^\star + \rho_-^\star = \gamma \rho_+ - \frac{\gamma^2 \rho_+}{\gamma} = 0. \tag{12}$$

Thus, the bulk charge density of a current-carrying wire vanishes in the rest frame of the conduction charges, rather than in the lab frame [25].

The positive charges experience a magnetic Lorentz force density in the $\star$ frame,

$$f_{+,r}^\star = -\frac{\rho_+^\star v B_\phi^\star}{c} = -\frac{2\pi \rho_+^{\star\,2} v^2 r}{c^2} = -\frac{2\pi \gamma^2 v^2 \rho_+^2 r}{c^2} = f_{+,r}. \tag{13}$$

This transverse Lorentz force is the same in the lab and the $\star$ frames, as expected for the transverse spatial component of a 4-vector.[5]

2.3   Galilean Electrodynamics

In sec. 2.2 we used Einstein's theory of special relativity [54], but in the 1800's it was more natural to assume that quantities in two different (inertial) frames were related by Galilean transformations. Of course, Galileo did not consider the transformation of electromagnetic fields between two frames, but only the transformation of the spacetime coordinates (x, y, z, t).

One consequence of Galilean relativity is that force vectors are invariant. So if the force on the conduction electrons vanishes in the lab frame, it also vanishes in the $\star$ frame of the conduction electrons in Galilean relativity. Hence, considerations of the internal force density in a current carrying wire in two different frames cannot distinguish special relativity from Galilean relativity.

While many workers in the 1800's considered aspects of the transformation of electromagnetic fields between (inertial) frame, assuming that Galilean relativity was valid here, it is surprising that a consistent view of this did not emerge until 1973 in an interesting paper by LeBellac and Levy-Léblond [53]. In this view there are no electromagnetic waves, and only quasistatic phenomena, so this notion is hardly compatible with Maxwellian electrodynamics,[6] but it can suffice for the present example.

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[5]The Lorentz force $\mathbf{f} = \rho \mathbf{E} + \mathbf{J}/c \times \mathbf{B}$ on a charge-current density 4-vector $j_\mu = (\rho, \mathbf{J}/c)$ is the spatial component of the 4-vector $f_{\text{Lorentz},\mu} = (\mathbf{J} \cdot \mathbf{E}/c, \mathbf{f})$. Thus, the Lorentz force density is "more relativistic" than the so-called Minkowski force based on the Newtonian relation $\mathbf{F} = m\mathbf{a}$, for which the 4-vector relation is $f_{\text{Minkowski},\mu} = (\gamma \mathbf{F} \cdot \boldsymbol{\beta}, \gamma \mathbf{F})$ for the force on a particle of velocity $\mathbf{v} = \boldsymbol{\beta} c$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$, in that the Minkowski force has the awkward factor of γ in its spatial components. Of course, the (Minkowski) force $\mathbf{F} = \mathbf{f}\,d\text{Vol}$ on a charged volume element, $d\text{Vol}$, is not a component of a 4-vector; however, $\gamma \mathbf{F} = \mathbf{f}\gamma\,d\text{Vol}$ is.

[6]It is argued in [55] that people should have deduced special relativity immediately after Maxwell's equations were written down (and without considerations of propagation of light).

In fact, there are two variants of Galilean electrodynamics, so-called electric Galilean relativity in which the transformations between two inertial frames with relative velocity $\mathbf{v}$ are (sec. 2.2 of [53], but given here in Gaussian units),[7]

$$\rho'_e = \rho_e, \qquad \mathbf{J}'_e = \mathbf{J}_e - \rho_e \mathbf{v} \qquad (c\,|\rho_e| \gg |J_e|),$$

$$\mathbf{E}'_e = \mathbf{E}_e, \qquad \mathbf{B}'_e = \mathbf{B}_e - \frac{\mathbf{v}}{c} \times \mathbf{E}_e \qquad \mathbf{f}_e = \rho_e \mathbf{E}_e \qquad \text{(electric)}, \quad (14)$$

and so-called magnetic Galilean relativity (sec. 2.3 of [53]) with transformations,

$$\rho'_m = \rho_m - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_m \qquad \mathbf{J}'_m = \mathbf{J}_m \qquad (c\,|\rho_e| \ll |J_e|),$$

$$\mathbf{E}'_m = \mathbf{E}_m + \frac{\mathbf{v}}{c} \times \mathbf{B}_m, \qquad \mathbf{B}'_m = \mathbf{B}_m \qquad \mathbf{f}_m = \rho_m \left(\mathbf{E}_m + \frac{\mathbf{v}}{c} \times \mathbf{B}_m \right) \quad \text{(magnetic)}. \ (15)$$

For comparison, the low-velocity limit of special relativity[8] has the transformations,

$$\rho'_s \approx \rho_s - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_s \qquad \mathbf{J}'_s \approx \mathbf{J}_s - \rho_s \mathbf{v},$$

$$\mathbf{E}'_s \approx \mathbf{E}_s + \frac{\mathbf{v}}{c} \times \mathbf{B}_s, \qquad \mathbf{B}'_s \approx \mathbf{B}_s - \frac{\mathbf{v}}{c} \times \mathbf{E}_s \qquad \text{(special relativity, } v \ll c\text{)}. \ (16)$$

Only electric Galilean relativity (14) applies to the present example, since the conduction current obeys $J = \rho_- v \ll \rho_- c$. Note that there is no $\mathbf{v} \times \mathbf{B}_e$ force in this case. Since electric charge density ρ_e and electric field $\mathbf{E}_e$ are invariant, so is the force density $\mathbf{f}_e$, and the force on the conduction electrons vanishes in both the lab frame and the * frame.

2.4 Clausius' Paradox

Clausius (1877) [1, 2] did not consider forces on the conduction electrons inside the wire, but rather the force on a charge q outside the wire and at rest with respect to it in the lab frame, arguing that this force must be zero.[9] It seems to this author that Clausius confused the requirement that the radial force on the conduction electrons inside the wire must be zero with the force on an external charge at rest, which latter force need not be zero. As such, he did not make the argument of sec. 2.1 that shows that the volume charge density inside a current carrying wire is nonzero. Instead, assuming that the force on an external charge must be zero, he postulated that the total charge density inside the wire is zero, $\rho_+ + \rho_- = 0$.[10] Perhaps the simplicity of this assumption convinced many people that it must be true, as it is often made.[11]

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[7]In Galilean electrodynamics the symbol c does not represent the speed of light (as light does exist in this theory), but only the function $1/\sqrt{\epsilon_0 \mu_0}$ of the (static) permittivity and permeability of the vacuum.

[8]Of course, electromagnetic waves can exist in special relativity, and propagate in vacuum with speed c.

[9]See the bottom of p. 86 of [1].

[10]This has been called the Clausius Postulate in [56]. A variation is the claim [33] that while the volume charge density ρ inside a wire can be nonzero, the total linear charge density of a wire must be zero in the lab frame; but the only physical requirement is that the total charge in/on the circuit be the same before and after the current is excited.

[11]A (surprising) example of the use of the Clausius Postulate is sec. 13-6 of [57]. This postulate is claimed to be "evident" after eq. (7.5), $\mathbf{\nabla} \cdot \mathbf{E} = \mathbf{\nabla} \cdot \mathbf{J}/\sigma = 0$, of [59], inside a conductor of conductivity σ with steady

Besides having a small nonzero internal charge density, current-carrying resistive wires have a surface charge density [35]. Here, we will follow Clausius (joining his many followers) in ignoring the effect of the surface charge density on the external charge q.

Then, assuming (electric) Galilean relativity for the transformation of the charge density to the * frame of the conduction electrons, the total charge density is zero in this frame as well as in the lab frame.

However, the positive charges are in motion in the * frame, and generate a magnetic field of the same strength as the field of the moving electrons in the lab frame, $\mathbf{B}^\star = \mathbf{B}$.[12]

So, if we believe that there exists a $\mathbf{v} \times \mathbf{B}$ force on charge q in the * frame, the wire exerts a nonzero force on this charge in this frame, but the force is zero in the lab frame. This is Clausius' paradox.[13]

It did not occur to anyone prior to 1973 that electric Galilean relativity [53], eq. (14), has no $\mathbf{v} \times \mathbf{B}$ force, which avoids the paradox from one perspective.

Budde (1880) [61, 62] surmised that the charge densities are not the same in the * frame and the lab frame, and that the wire has a "compensation charge" in that frame sufficient to bring the Lorentz force to zero. This suggestion is a precursor of the argument of special relativity (given in sec. 13-6 of [57] for the assumption that $\rho_+ + \rho_- = 0$ in the lab frame), as acknowledged by Lorentz on p. 1242 of [63]. However, we argue that there is actually a small nonzero electromagnetic force (of order v^2/c^2) on the external charge in both the lab frame and in the * frame, related according to special relativity.[14]

The ingredients of Clausius' paradox were mostly in place in 1865 in Maxwell's *Dynamical Theory of the Electromagnetic Field* [67]: the Biot-Savart force density, $\mathbf{f} = \rho \mathbf{E} + \mathbf{J}/c \times \mathbf{B}$, and the time-independent electromagnetic equations, $\nabla \cdot \mathbf{E} = 4\pi\rho$, $\nabla \times \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$, and $\nabla \times \mathbf{B} = 4\pi\mathbf{J}/c$. These have the implication that "static" electricity and magnetism involving currents with charge motion at velocity $\mathbf{v}$ include effects of order v^2/c^2 in the lab frame,[15] and that a consistent view of these effects in a moving frame (such as the rest frame of the moving charges) requires special relativity rather than Galilean relativity. Yet, it took 40 years for these consequences to be appreciated fully.

It is now sometimes said that electricity plus special relativity imply magnetism, but a more historical view is that (static) electricity plus magnetism implies special relativity.

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current density $\mathbf{J}$. However, the magnetic field inside the conductor should be included in the generalized Ohm's law (due to Hall (1879) [50]), $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$, such that $0 = \nabla \cdot \mathbf{J}/\sigma = \nabla \cdot \mathbf{E} - \mathbf{v} \cdot \nabla \times \mathbf{B}$. Then, the charge density is related by $\rho = \mathbf{v} \cdot \mathbf{J}/c^2 = \rho_- v^2/c^2$ as found in eq. (6).

The Clausius postulate was also assumed in sec. 5.9 of [58].

[12]This is consistent with the field transformations listed in eq. (14).

[13]This paradox was actually posed by Fröhlich [3, 4], in the context of the debate between Clausius and Weber as to whether electrical currents involve motion of both positive and negative charges (Weber [60]), or charges of only one sign (Clausius [1]). For an extensive discussion of the themes of this note, including the Clausius-Weber debate, see [42].

[14]In practice, the necessary surface charges on the current-carrying wire also exert a small force on the external charge, so it is not easy to distinguish the two components of the force in experiments [64, 65, 66].

[15]Discussion of electricity and magnetism in the early 1800's was decoupled from the issue of whether the surface of the Earth was a rest frame for the æther that supported propagation of light.

2.5 Circular Loop of Current

The preceding discussion has tacitly assumed that any curvature of the wire can be ignored. For conduction electrons at radius r from the center of a circular loop of wire, the radial electromagnetic force must not be zero, as assumed above, but rather mv^2/r, where m is the (relativistic) mass of the electron, with this force pointing towards the center of the loop. Some azimuthal asymmetry in the current density and/or in the surface charge density is needed to provide this centripetal force.

A   Appendix: Currents of Both Signs

The preceding discussion has tacitly assumed that electrical current in resistive conductors consists of the flow of only one sign of charged particles, namely, electrons. This was not clear until Hall's work in 1879 [50], and Weber's electrodynamics [42] assumed that the current consisted of flow of charges of both signs.[16]

Assuming the charge inside the wire consists of only two species with charge densities $\rho_+(r)$ and $\rho_-(r)$ inside the wire with z-velocities v_+ and v_- independent of radius r, the radial electric field inside the wire is still given by eq. (2) while the azimuthal magnetic field is now,

$$B_\phi(r) = \frac{4\pi}{cr} \int_0^r r'[v_+\rho_+(r') + v_-\rho_-(r')]\, dr'. \tag{17}$$

The radial component of the Lorentz force density on the negative charges is,

$$f_{-,r} = \frac{4\pi\rho_-(r)}{r} \int_0^r r' \left[\rho_+(r') + \rho_-(r') - \frac{v_-v_+\rho_+(r')}{c^2} - \frac{v_-^2\rho_-(r')}{c^2} \right] dr', \tag{18}$$

and that on the positive charges is,

$$f_{+,r} = \frac{4\pi\rho_+(r)}{r} \int_0^r r' \left[\rho_+(r') + \rho_-(r') - \frac{v_+^2\rho_+(r')}{c^2} - \frac{v_+v_-\rho_-(r')}{c^2} \right] dr'. \tag{19}$$

For steady axial motion both radial forces must separately vanish at all r inside the wire, which implies that,

$$\rho_+(1 - v_+v_-/c^2) = -\rho_-(1 - v_-^2/c^2), \qquad \rho_+(1 - v_+^2/c^2) = -\rho_-(1 - v_+v_-/c^2), \tag{20}$$

$$(1 - v_+v_-/c^2)^2 = (1 - v_-^2/c^2)(1 - v_+^2/c^2), \tag{21}$$

$$v_+^2 - 2v_+v_- + v_-^2 = 0, \tag{22}$$

$$v_+ = v_-, \tag{23}$$

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[16]There was a long tradition of the assumption of currents of both signs, including Oersted [68], Ampère [69] and Fechner [70]. Clausius (1877) [1] was perhaps the first to go against this trend.

and then eq. (20) tells us that,

$$\rho_- = -\rho_+, \qquad \rho = \rho_+ + \rho_- = 0, \qquad J_z = \rho_+ v_+ + \rho_- v_- = 0. \tag{24}$$

Thus, it appears that a model of steady electrical current as involving flow of two species of charged particles (both of which are in motion) is inconsistent, if the only radial force on the currents is electromagnetic.[17,18]

B   Appendix: Square Current Loop

This Appendix added Oct. 22, 2020, based on a comment by G. D'Abramo.

We discuss how charge conservation applies to a square loop of electric current, as viewed in both its rest frame and in a frame where the loop has velocity **v** parallel to two of its side. We neglect the surface charge on the wire.

Suppose that in the rest frame of the loop each of its four sides (of rest length L) has positive lattice charge q_+. Then if the speed of the conduction electrons in the rest frame is u, the total charge of the conduction electrons in each of the four sides is (in the rest frame of the loop),

$$q_- = -q_+(1 - u^2/c^2), \tag{25}$$

recalling eq. (5).

In any inertial frame, the positive charge observed on each of the four sides of the loop (which does not appear to be square in general) is q_+. And in any inertial frame the total negative charge on all four sides of the loop is $4q_-$. However, the negative charge on a particular side of the loop is not, in general, q_-.

We illustrate this for observations of the charge on the sides of the loop at time $t' = 0$ in the $'$ frame where the loop has velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$. The loop has its corners at $(x, y) = (0, 0)$, $(0, L)$, $(L, 0)$ and (L, L) in its rest frame. We also suppose that the velocity of the conduction electrons in the side from $(0, 0)$ to $(0, L)$ is $\mathbf{u} = u\,\hat{\mathbf{x}}$.

The coordinates of points in the $'$ frame are related to those in the rest frame of the loop by the Lorentz transformations,

$$x' = \gamma(x - vt), \qquad y' = y, \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right), \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{26}$$

Then, at $t' = 0$, points with $x = L$ correspond to,

$$t = \frac{vL}{c^2}, \qquad x' = \gamma\left(L - \frac{v^2}{c^2}L\right) = \frac{L}{\gamma}, \tag{27}$$

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[17]We could also regard the symbols $+$ and $-$ to represent two flows of the same sign of charge with different charge densities and speeds, and conclude that this is not possible.

[18]If $\rho_- = -\rho_+$ and $v_- = -v_+$, then $f_{-,r} = f_{+,r} < 0$. So, if the moving charges behave like fluids that can support a pressure with a radial gradient that generates an outward force on the fluid currents, the two-species model can be consistent.

and in the $'$ frame the sides parallel to the x-axis (and to $\mathbf{v}$) appear to be Lorentz contracted (with lenght L/γ, while the sides perpendicular to $\mathbf{v}$ appear to have the uncontracted length L.

Since all points on each of the sides perpendicular to $\mathbf{v}$ at time $t' = 0$ correspond to a single time in the rest frame, these side are observed in the $'$ frame to have the same negative charge as in the rest frame at a fixed time,

$$q' = q_- \qquad \text{[sides from } (x, y) = (0, 0) \text{ to } (0, L) \text{ and } (L, 0) \text{ to } (L, L)]. \tag{28}$$

However, at time $t' = 0$ different points on the sides parallel to $\mathbf{v}$ correspond to different times in the rest frame, so the negative (moving) charge in these sides is different from $q-$.

In particular, the point $(x, y) = (0, 0)$ corresponds to $(x', y') = (0, 0)$ when $t' = 0$, so an electron at this point at time $t = 0$ is also the electron at the point $(x', y') = (0, 0)$ at time $t' = 0$. However, the point $(x, y) = (L, 0)$ corresponds to $(x', y') = (L/\gamma, 0)$ and time $t = vL/c^2$ when $t' = 0$. Thus, the moving, negative charge at this point at time $t' = 0$ was at $x = L - ut = L(1 - uv/c^2)$ in the rest frame at time $t = 0$, recalling that the velocity of the charge in the side from $(0, 0)$ to $(L, 0)$ is $u\,\hat{\mathbf{x}}$ in the rest frame. Hence, the charge observed in the $'$ frame at $t' = 0$ in this side is only,

$$q' = q_- \left(1 - \frac{uv}{c^2}\right) \qquad \text{[side from } (x, y) = (0, 0) \text{ to } (L, 0)]. \tag{29}$$

In contrast, the velocity of the charge in the side from $(0, L)$ to (L, L) is $-u\,\hat{\mathbf{x}}$ in the rest frame, so the charge at the point $(x', y') = (L/\gamma, L)$ at $t' = 0$ was at $(x, y) = (L, L - uvL/c^2$ at time $t = 0$, and the charge observed in the $'$ frame in this side is greater than that in the rest frame,

$$q' = q_- \left(1 + \frac{uv}{c^2}\right) \qquad \text{[side from } (x, y) = (0, L) \text{ to } (L, L)]. \tag{30}$$

The total negative charge observed in the $'$ frame at time $t' = 0$ is,

$$q'_{-,\text{total}} = 4q_-, \tag{31}$$

consistent with conservation of total negative charge.

We recognize this example as showing how a current loop/magnetic dipole appears to have an electric dipole moment in a frame where the magnetic dipole is in motion. This was first discussed by Lorentz in 1910 [63].[19]

In detail, we note that the electric current in the lab frame is $I = q_- u/L$, and the lab-frame magnetic moment of the current loop is,

$$\mathbf{m} = \frac{I\mathbf{A}}{c} = \frac{IL^2}{c}\,\hat{\mathbf{z}}. \tag{32}$$

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[19]That a moving current leads to an apparent charge separation was noted in [61] (1880). Considerations of the "motional" electric dipole, $\mathbf{p} \approx \mathbf{v}/c \times \mathbf{m}_0$ when $\mathbf{p}_0 = 0$, due the motion of a magnetic dipole appears, in eq. (17) of [71]. See also eq. (13) of [72], which was inspired by Art. 600 of [73]. This effect was also discussed by Frenkel [74, 75] following the prediction by Uhlenbeck and Goudsmit [76] that an electron has an intrinsic magnetic moment. For additional comments by the author, see [77].

In the $'$ frame the electric dipole moment of the loop appears to be,

$$\mathbf{p}' = \frac{q'uvL}{c^2}\,\hat{\mathbf{y}} = \frac{IL^2v}{c^2}\,\hat{\mathbf{y}} = \frac{mv}{c}\,\hat{\mathbf{y}} = \mathbf{m} \times \frac{\mathbf{v}}{c}\,, \tag{33}$$

for $\mathbf{v} = v\,\hat{\mathbf{x}}$.

References

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