The Electromagnetic Fields Outside a Wire That Carries a Linearly Rising Current

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1 Problem

A neutral wire along the z-axis carries current I that varies with time t according to,

$$I(t) = \begin{cases} 0 & t \leq 0, \\ \alpha t & t > 0, \qquad \alpha \text{ is a constant.} \end{cases} \tag{1}$$

Deduce the time-dependence of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, observed at a point $(r, \theta = 0, z = 0)$ in a cylindrical coordinate system about the wire. Use your expressions to discuss the fields in the two limiting cases that $ct \gg r$ and $ct = r + \epsilon$, where c is the speed of light and $\epsilon \ll r$.

2 Solution

We follow the familiar method of first calculating the retarded potentials and then taking derivatives to find the fields. The retarded scalar and vector potentials V and $\mathbf{A}$ are given by,

$$V(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t - R/c)\, d^3\mathbf{x}'}{R}, \qquad \text{and} \qquad \mathbf{A}(\mathbf{x}, t) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x}', t - R/c)\, d^3\mathbf{x}'}{R}, \tag{2}$$

in Gaussian units, where ρ and $\mathbf{J}$ are the charge and current densities, respectively, and $R = |\mathbf{x} - \mathbf{x}'|$.

In the present case, we assume that the wire remains neutral when the current flows.[1] Then, the scalar potential vanishes. For the vector potential, we see that only the component A_z will be nonzero. Also, $\mathbf{J}\, d^3\mathbf{x}'$ can be rewritten as $I\, dz$ for current in a wire along the z-axis. For an observer at $(r, 0, 0)$ and a current element at $(0, 0, z)$, we have $R = \sqrt{r^2 + z^2}$. Further, the condition that I is nonzero only for time $t > 0$ implies that it contributes to the fields only for z such that $(ct)^2 > R^2 = r^2 + z^2$. That is, we need to evaluate the integral only for,

$$|z| < z_0 \equiv \sqrt{(ct)^2 - r^2}. \tag{3}$$

Altogether,

$$A_z(r, 0, 0, t) = \frac{\alpha}{c} \int_{-z_0}^{z_0} \left(\frac{t}{\sqrt{r^2 + z^2}} - \frac{1}{c} \right) dz = \frac{\alpha}{c} \left(t \ln \frac{ct + z_0}{ct - z_0} - \frac{2z_0}{c} \right) = \frac{2\alpha}{c} \left(t \ln \frac{z_0 + ct}{r} - \frac{z_0}{c} \right). \tag{4}$$

[1]We ignore the small departure from neutrality that scales as v^2/c^2 where $v \approx 1$ cm/sec is the velocity of the conduction electrons. See, for examlple, [1].

[The two forms in eq. (4) arise depending on whether or not one notices that the integrand is even in z.]

The magnetic field is obtained via $\mathbf{B} = \nabla \times \mathbf{A}$. Since only A_z is nonzero, the only nonzero component of $\mathbf{B}$ is (noting that $\partial z_0/\partial r = -r/z_0$),

$$B_\phi = -\frac{\partial A_z}{\partial r} = \frac{2\alpha z_0}{cr}. \tag{5}$$

The only nonzero component of the electric field is,

$$E_z = -\frac{1}{c}\frac{\partial A_z}{\partial t} = -\frac{2\alpha}{c^2}\ln\frac{z_0 + ct}{r}. \tag{6}$$

For long times, $ct \gg r$, $\Rightarrow z_0 \approx ct$, and the fields become,

$$B_\phi \approx \frac{2\alpha t}{cr} = \frac{2I(t)}{cr} = B_0(t), \qquad E_z \approx -\frac{2\alpha}{c^2}\ln\frac{2ct}{r} = -B_0\frac{r}{ct}\ln\frac{2ct}{r} \ll B_0, \tag{7}$$

where $B_0(t) = 2I(t)/cr$ is the instantaneous magnetic field corresponding to current I(t). That is, we recover the magnetostatic limit at large times.

For short times, $ct = r + \epsilon$ with $\epsilon \ll r$, after the fields first become nonzero we have,

$$z_0 = \sqrt{2r\epsilon + \epsilon^2} \approx \sqrt{2r\epsilon}, \tag{8}$$

so,

$$B_\phi \approx \frac{2\alpha}{c^2}\sqrt{\frac{2\epsilon}{r}}, \qquad \text{and} \qquad E_z \approx -\frac{2\alpha}{c^2}\ln\frac{r + \epsilon + \sqrt{2r\epsilon}}{r} \approx -\frac{2\alpha}{c^2}\sqrt{\frac{2\epsilon}{r}} = -B_\phi. \tag{9}$$

In this regime, the fields have the character of radiation, with $\mathbf{E}$ and $\mathbf{B}$ of equal magnitude, mutually orthogonal, and both orthogonal to the line of sight to the closest point on the wire. (Because of the cylindrical geometry, the radiation fields do not have $1/r$ dependence,vwhich holds instead for cylindrical static fields.)

In sum, the fields build up from zero only after time $ct = r$. The initial fields propagate outwards at the speed of light and have the character of cylindrical waves. But at a fixed transverse distance r, the electric field dies out with time, and the magnetic field approaches the instantaneous magnetostatic field due to the current in the wire.[2,3]

[2] Of possible amusement is a direct calculation of the vector potential for the case of a constant current I_0. First, from Ampere's law we know that $B_\phi = 2I_0/cr = -\partial A_z/\partial r$, so we have that,

$$A_z = -\frac{2I_0}{c}\ln r + \text{const.} \tag{10}$$

If we use the integral form for the vector potential we have,

$$A_z(r,0,0) = \frac{1}{c}\int_{-\infty}^{\infty}\frac{I_0 dz}{\sqrt{r^2+z^2}} = \frac{2I_0}{c}\int_0^{\infty}\frac{dz}{\sqrt{r^2+z^2}} = -\frac{2I_0}{c}\ln r + \lim_{z\to\infty}\ln(z + \sqrt{z^2+r^2}). \tag{11}$$

Only by ignoring the large constant, which does not depend on r for a long wire, do we recover the "elementary" result.

[3] For the related example of the fields associated with a linearly rising current in a solenoid, see [2].

References

[1] K.T. McDonald, *Charge Density in a Current-Carrying Wire*, (Dec. 23, 2010), http://kirkmcd.princeton.edu/examples/wire.pdf

[2] K.T. McDonald, *The Fields Outside a Long Solenoid with a Time-Dependent Current*, (Dec. 6, 1996), http://kirkmcd.princeton.edu/examples/solenoid.pdf