Wireless Power Transmission

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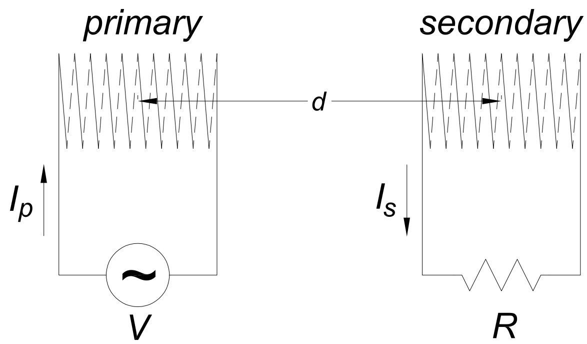
1 Problem

Discuss wireless power transmission between a pair of electrical circuits that contain coils.



2 Solution

The related issue of the flow of energy in such circuits as described by the Poynting vector is considered in [1].

Supposing the distance d between the centers of the two coils is small compared to the wavelength $\lambda = 2\pi c/\omega$, where c is the speed of light in vacuum, circuit analysis is a good approximation. Writing the self inductance of the two coils as $L_p = L_s = L$ and their mutual inductance as $M(d)$, the circuit equations are (for time dependence $e^{i\omega t}$ for the voltage source),

$$V = I_p Z_p + i\omega M I_s, \qquad Z_p = i\omega L, \qquad 0 = I_s Z_s + i\omega M I_p, \qquad Z_s = R + i\omega L. \tag{1}$$

Then,

$$I_s = -\frac{i\omega M}{Z_s} I_p, \qquad I_p = V \frac{Z_s}{Z_p Z_s + \omega^2 M^2}\,, \qquad I_s = -V \frac{i\omega M}{Z_p Z_s + \omega^2 M^2}\,, \tag{2}$$

and the (time-average) power delivered to the load resistor R is,

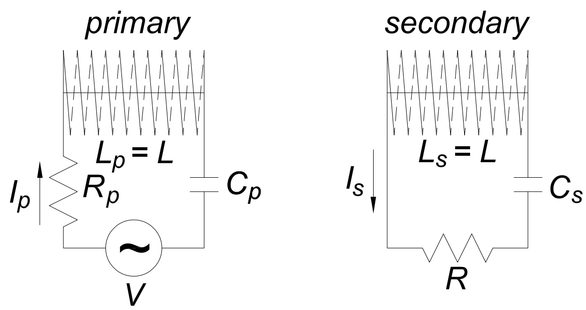
$$P_R = \frac{|I_s|^2 R}{2} = \frac{V^2}{2} \frac{\omega^2 M^2 R}{|Z_p Z_s + \omega^2 M^2|^2}\,. \tag{3}$$

For the circuits shown above, with impedances Z_p and Z_s as stated in eq. (1), the delivered power would be,

$$P_R = \frac{V^2}{2} \frac{\omega^2 M^2 R}{|i\omega L R - \omega^2 L^2 + \omega^2 M^2|^2} = \frac{V^2}{2R} \frac{M^2}{L^2[1 + \omega^2 (L^2 - M^2)^2 / L^2 R^2]}. \tag{4}$$

In general, $M^2 \le L_p L_s = L^2$, and $M(d) \propto 1/d^3$ for coil separations d large compared to the coil size (see the Appendix). Hence, the (idealized) circuits on p. 3, for which $P_R \propto 1/d^6$, are not favorable for wireless transmission of energy.

To maximize the delivered power (3) we should minimize Z_p or Z_s or both. For example, a capacitor C_p and a small resistor R_p could be added in series to the coil in the primary circuit,[1] such that this series R-L-C circuit resonates at angular frequency ω, with $Z_p = R_p$.[2] Similarly, a capacitor C_s could be added in series in the secondary circuit such that it also resonates at angular frequency ω, with $Z_s = R$.



Then,

$$P_R = \frac{V^2}{2} \frac{\omega^2 M^2 R}{(R_p R + \omega^2 M^2)^2} . \tag{5}$$

Although the mutual inductance gets smaller rapidly with increasing distance d between the coils, it is possible that $R_p R \ll \omega^2 M^2$ for a useful range of d. If so,

$$P_R \approx \frac{V^2}{2} \frac{R}{\omega^2 M^2} = \frac{V^2}{2R} \left(\frac{R}{\omega M} \right)^2 \qquad (R_p R \ll \omega^2 M^2), \tag{6}$$

and the power delivered to the load resistor R could increase with distance (at least for $r \ll \lambda$)!

Of course, power is also dissipated in the resistor R_p, although this is small if $R_p R \ll \omega^2 M^2$,

$$P_{R_p} = \frac{|I_p|^2 R_p}{2} = \frac{V^2}{2} \frac{|Z_s|^2 R_p}{|Z_p Z_s + \omega^2 M^2|^2} = \frac{V^2}{2} \frac{R^2 R_p}{(R R_p + \omega^2 M^2)^2} \approx \frac{V^2}{2} \frac{R}{\omega^2 M^2} \frac{R R_p}{\omega^2 M^2} \ll P_R. \tag{7}$$

Then, if one defines the power transmission efficiency as,

$$\epsilon = \frac{P_R}{P_R + P_{R_p}} = \frac{\omega^2 M^2}{\omega^2 M^2 + R R_p} , \tag{8}$$

[1] Any real inductor has some capacitance and resistance; C_p and R_p are the total capacitance and resistance in the primary circuit, including that of the nominal inductor.

[2] Strictly, the analysis should include effects of radiation by the circuits, which are small loop antennas. To a first approximation, this affects the impedance of the circuit by the addition of a (series) "radiation resistance" $R_{\rm rad} = 31171 (N \mathrm{Area}/\lambda^2)^2\,\Omega$, which is of order $1\,\Omega$ for typical geometries relevant to wireless power transmission. That is, the resistances R_p and R include the radiation resistance, and so cannot be less than $\approx 1\ \Omega$.

this efficiency could be close to unity when $RR_p \ll \omega^2 M^2$.

While inductors are typically thought of as solenoidal coils (with length b many times their radius a), it is considered more practical for wireless power transmission to use planar spiral coils of outer radius a (and inner radius a' that we take to be zero in this note). Either type of coil has a self capacitance $C_{\rm self}$ between adjacent turns in the winding, and it is convenient to operate the system at the natural (angular) frequency $\omega_0 = 1/\sqrt{LC_{\rm self}}$, where L is the self inductance of the coils (taken here to be the same for both coils).

For a numerical estimate, suppose that $R = 10\,\Omega$, $R_p \approx 1\,\Omega$, and that the planar spiral coils have $N = 700$ turns of outer radius $a = 0.3$ m. Then, as estimated in Appendix A.2, $(\omega_0 M)^2 = 10(3/d)^6\ \Omega^2$ for coil separation d in meters. In this case, $\omega_0 \approx 10$ kHz, and $f_0 = 1600$ Hz. Then, according to eqs. (5) and (8),

$$P_R \approx \frac{V^2}{2R}\frac{10(3/d)^6}{[1+(3/d)^6]^2} \qquad \epsilon \approx \frac{(3/d)^6}{1+(3/d)^6}\,, \tag{9}$$

which is maximal for $d = 3$ m where the efficiency is $\epsilon \approx 0.5$. The efficiency at, say, $d = 2$ m would be $\approx 92\%$.[3]

Considerations such as the above have led to optimism of practical wireless power transmission, famously starting with Tesla [2]-[7] (who noted the importance of resonant circuits, but advocated high voltage to compensate for low ωM). An analysis similar to that given here appears in [8], where Fig. 4 shows an example of the transferred power first increasing with distance and then falling at larger distance; subsequent efforts include methods to extend the range of "good" distances by sensing the location of the load circuit and electronically adapting the parameters of the circuits for optimal efficiency at that distance [9]. Among the vast literature, see also [10]-[12].

3 High Frequencies

The preceeding discussion assumed that the wavelength $\lambda = 2\pi c/\omega$ was large compared to all relevant length scales in the coupled-circuit problem. Considerations of circuits of size of order the operating wavelength arise for antenna arrays (see, for example, [13]), and later became important with the development of integrated circuits for which appropriate modifications to circuit analysis were pioneered by Ruehli [14], and are incorporated into integrated-circuit simulations such as SPICE [15].

In this section we consider the case that the coil separation d is of order λ, but that the primary and secondary circuits are each small compared to λ. For wireless power transmission in "room-size" applications, the operating frequency would be in excess of 100 MHz.

At such high frequencies, radiation by the circuits is not entirely negligible. Here, we simply suppose that the resistances R_p and R include the radiation resistances of the primary and secondary circuits; for practical power transmission the radiation resistance of the secondary should be small compared to the Ohmic resistance of the load in that circuit.

[3]However, if we desire $(\omega_0 M)^2 = RR_p$ at larger coil separation d, the number of turns required grows rapidly with d according to eq. (46) (and the self-resonant frequency ω_0 drops rapidly, according to eq. (45). This suggests that efficient wireless power transmission via coupled coils will be restricted to relatively small distances.

If we suppose that coupled-circuit analysis of the type used above can be applied to the high-frequency case, we are led to a paradox (noted in [16, 17]), that if the coil separation is $\lambda/4$ then the back reaction of the secondary on the primary is shifted in phase by 180° compared to the case of small separation such that the back reaction reinforces the drive voltage, perhaps leading to "infinite" currents in the circuit.[4]

[4] A naïve analysis is as follows.

The separation d between the primary and secondary coils implies that wave propagating across the space between these circuits accumulate a phase retardation,

$$\phi = kd = \frac{2\pi d}{\lambda} = \frac{\omega d}{c}\,. \tag{10}$$

The revised circuit equations are,

$$V\,e^{i\omega t} = I_p\,e^{i\omega t} Z_p + i\omega M I_s\,e^{i\omega t'}, \qquad 0 = I_s\,e^{i\omega t} Z_s + i\omega M I_p\,e^{i\omega t'}, \tag{11}$$

in that the $\mathcal{EMF}$ in the primary at time t due to the coupling with the secondary depends on the current in the latter at the earlier (retarded) time $t' = t - d/c = t - \phi/\omega$ (and similarly the $\mathcal{EMF}$ in the secondary at time t depends on the current in the primary at the retarded time t'). Dividing out the common factors $e^{i\omega t}$, as usual, we obtain,

$$V = I_p Z_p + i\omega M I_s\,e^{-i\phi}, \qquad 0 = I_s Z_s + i\omega M I_p\,e^{-i\phi}, \tag{12}$$

Then,

$$I_s = -\frac{i\omega M\,e^{-i\phi}}{Z_s} I_p, \qquad I_p = V\frac{Z_s}{Z_p Z_s + \omega^2 M^2\,e^{-2i\phi}}\,, \qquad I_s = -V\frac{i\omega M\,e^{-i\phi}}{Z_p Z_s + \omega^2 M^2\,e^{-2i\phi}}\,. \tag{13}$$

Again assuming that the primary and secondary are each resonant circuits at angular frequency ω, with $Z_p = R_p$ and $Z_s = R$, we have,

$$I_p = V\frac{R}{R_p R + \omega^2 M^2\,e^{-2i\phi}}\,, \qquad I_s = -V\frac{i\omega M\,e^{-i\phi}}{R_p R + \omega^2 M^2\,e^{-2i\phi}}\,. \tag{14}$$

If the coil separation is $d = \lambda/4$, then $\phi = \pi/2$ $(= 90°)$, and,

$$I_p = V\frac{R}{R_p R - \omega^2 M^2}\,, \qquad I_s = -V\frac{\omega M}{R_p R - \omega^2 M^2} \qquad (d = \lambda/4). \tag{15}$$

In the special case that $R_p R = \omega^2 M^2$, the currents are formally infinite (which is a clue that the analysis is too naïve).

An additional difficulty with the above analysis concerns energy conservation. The (time-average) power delivered by the voltage source is,

$$P_{\rm source} = \frac{V I_p}{2} = V^2 \frac{R}{2(R_p R - \omega^2 M^2)}\,, \tag{16}$$

while the power consumed by the load resistors is,

$$P_p + P_s = \frac{I_p^2 R_p}{2} + \frac{I_s^2 R}{2} = V^2 R \frac{R_p R + \omega^2 M^2}{2(R_p R - \omega^2 M^2)^2}\,. \tag{17}$$

These are equal only if the factor in the denominator were $R_p R + \omega^2 M^2$, corresponding to coil separation $d \ll \lambda$. We infer that the above type of coupled circuit analysis is inappropriate for any coil separation that is a significant fraction of a wavelength, and that more advanced analyses are required in such cases.

The case of coil separation being a significant fraction of a wavelength requires use of techniques of antenna analysis, which are best performed numerically, using codes such as NEC4 [18] that solve an appropriate integral equation.[5]

The present example is called a two-element loop Yagi in the antenna community, where Yagi antennas more typically have linear elements. Analytic discussion of Yagi antennas is given, for example, in [13, 20]. The coupled-circuit analysis is based on the equations,

$$V = I_p Z_p + I_s Z_M, \qquad 0 = I_s Z_s + I_p Z_M, \tag{18}$$

where the source voltage V and currents I_p and I_s all have time dependence $e^{i\omega t}$, Z_p and Z_s are the (self) impedances of the primary and secondary circuits, and $Z_M = Z_{ps} = Z_{sp}$ is the mutual impedance between the circuits. The mutual impedance also arises in the context of antenna reciprocity theorems [21, 22, 23]. The mutual impedance can be computed according to eq. (22.3.4) of [20],

$$Z_M = -\frac{1}{I_p(0) I_s(0)} \oint_s E_{\parallel,ps} I_s \, dl_s, \tag{19}$$

where $E_{\parallel,ps}$ is the tangential component of the electric field from the primary circuit along the conductor of the secondary circuit.

For simplicity, we suppose both coils have N turns of radius a and their centers are separated by distance $d \gg a$, with the coils being coaxial. Then, the electric field due to current I_p in the primary coil is azimuthal, with value at the secondary coil,[6]

$$E_{\parallel,ps} = \frac{\mu_0 c}{4\pi} N \pi I_p a^2 \frac{a}{d} \left(\frac{k^2}{d} - \frac{ik}{d^2} \right) e^{-ikd}. \tag{20}$$

For small coils the current is independent of position in the coil, so the integral (19) (over the N turns of the secondary coil) reduces to,

$$Z_M = i\omega \frac{\mu_0 N^2 a^4}{2\pi d^3} (1 - ikd) \, e^{-ikd} = i\omega M (1 - ikd) \, e^{-ikd}, \tag{21}$$

where $k = \omega/c$ and $M = \mu_0 N^2 a^4 / 2\pi d^3$ is the mutual inductance at low frequency (where $kd \ll 1$).[7,8]

The circuit equations (18) have the solutions,

$$I_p = \frac{V Z_s}{Z_p Z_s - Z_M^2}, \qquad I_s = -I_p \frac{Z_M}{Z_s} = -\frac{V Z_M}{Z_p Z_s - Z_M^2}, \tag{22}$$

[5]See, for example, [19].

[6]See, for example, sec. 9.23 of [24].

[7]The DC mutual inductance M, usually associated with magnetism, was obtained via eq. (19) which explicitly involves only the electric field, thereby illustrating the unity of the electromagnetic field.

[8]The result (21) differs from that of eq. (12) by the factor $1 - ikd$. For coil separations $d \gtrsim \lambda$, where each coil is in the far zone of the other, the term kd leads the mutual impedance to vary as $1/d^2$ rather than $1/d^3$ as for small d. This behavior indicates that the coupling of the coils in the far zone is via the radiation fields rather than the quasistatic magnetic dipole field as latter holds for small d.

The (self) impedances of the circuits are real in case of resonance,

$$Z_p = R_p + R_{\rm rad}, \qquad Z_s = R + R_{\rm rad}, \tag{23}$$

where the radiation resistance of the small coils is,[9]

$$R_{\rm rad} = \frac{8Z_0\pi^5 N^2 a^4}{3\lambda^4} = 31171 \left(\frac{N\pi a^2}{\lambda^2}\right)^2 \Omega, \qquad \text{with} \qquad Z_0 = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega. \tag{24}$$

The (time-average) power delivered to the load resistor R in the secondary circuit is,

$$P_R = \frac{|I_s|^2 R}{2} = \frac{V^2 |Z_M|^2 R}{2\,|Z_p Z_s - Z_M^2|^2}. \tag{25}$$

The (time-average) powers dissipated in the primary and secondary circuits are, according to the circuit analysis [noting that Z_p and Z_s are real for resonant circuits according to eq. (23)],

$$P_p = \frac{|I_p|^2 Z_p}{2} = \frac{V^2 Z_s^2 Z_p}{2\,|Z_p Z_s - Z_M^2|^2}, \qquad P_s = \frac{|I_s|^2 Z_s}{2} = \frac{V^2 |Z_M|^2 Z_s}{2\,|Z_p Z_s - Z_M^2|^2}. \tag{26}$$

We define the efficiency of the power transfer as,

$$\begin{aligned} \epsilon &= \frac{P_R}{P_p + P_s} = \frac{R\,|Z_M|^2}{Z_s(Z_p Z_s + |Z_M|^2)} \\ &= \frac{\omega^2 M^2 (1 + k^2 d^2) R}{(R + R_{\rm rad})[(R_p + R_{\rm rad})(R + R_{\rm rad}) + \omega^2 M^2 (1 + k^2 d^2)]} < 1, \end{aligned} \tag{27}$$

recalling eq. (23). High efficiency is achieved only if $R \gg R_{\rm rad}$ and $RR_p \ll \omega^2 M^2 (1+k^2 d^2)$.[10]

[9] The radiation resistance can be related to the reactance $X_{\rm coil} = \omega L = \mu_0 \omega N^2 \pi a^2 / b$ of the coil by $R_{\rm rad} = (4\pi^3/3)({\rm Vol}/\lambda^3) X_{\rm coil}$, where the volume of the coil is $\pi a^2 b$.

[10] The present circuit analysis is not fully consistent in that the power delivered by the source,

$$P_{\rm source} = \frac{Re(VI_p)}{2} = Re\left(\frac{V^2 Z_s}{2(Z_p Z_s - Z_M^2)}\right) = \frac{V^2 Z_s (Z_p Z_s - Re Z_M^{\star 2})}{2\,|Z_p Z_s - Z_M^2|^2}, \tag{28}$$

is not quite equal to the sum of the dissipated powers, eq. (26). One aspect of this inconsistency is that the power radiated by the two circuits is subject to interference effects, and is not simply the sum of the powers radiated by each circuit separately, as assumed in the circuit analysis.

In more detail, the electric field in the far zone of the two coils (*i.e.*, for $r \gg \lambda$ and $r \gg d$) is,

$$E_{{\rm rad},\phi} = \frac{\mu_0 c}{4\pi} N\pi a^2 k^2 (I_p + I_s\, e^{-ikd\cos\theta}) \sin\theta \frac{e^{ikr}}{r} = \frac{Z_0 N \pi^2 a^2}{\lambda^2} (I_p + I_s\, e^{-ikd\cos\theta}) \sin\theta \frac{e^{ikr}}{r}. \tag{29}$$

The (time-average) angular distribution of the radiated power at large distance r from the system is related to the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ by,

$$\frac{dP_{\rm rad}}{d\Omega} = r^2 \langle S_r \rangle = \frac{r^2 |E_{{\rm rad},\phi}|^2}{2Z_0} = \frac{1}{2} \frac{Z_0 \pi^4 N^2 a^4}{\lambda^4} \left[|I_p|^2 + |I_s|^2 + 2Re(I_P^\star I_s\, e^{-ikd\cos\theta})\right] \sin^2\theta, \tag{30}$$

A Appendix: Estimates of Inductances and Self Capacitance

A.1 Solenoidal Coils

In the lowest approximation, a coil of N turns of radius a and total length $b \gg a$ that carries current I has internal magnetic field $B = \mu_0 NI/b$ and self-flux $\Phi = LI = N\pi a^2 B = \mu_0 N^2 \pi a^2 I/b$, such that the self inductance is

$$L \approx \frac{\mu_0 N^2 \pi a^2}{b}\,, \tag{32}$$

and its magnetic moment has magnitude $m = N\pi a^2 I$. The magnetic field outside the coil at distance $\mathbf{r}$ is approximately,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} = \frac{\mu_0 N\pi a^2 I}{4\pi}\frac{3(\hat{\mathbf{m}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{m}}}{r^3}\,. \tag{33}$$

The magnetic flux from coil 1 that is linked by coil 2 is,

$$\Phi_{12} = M_{12} I_1 \approx N\pi a^2\, \hat{\mathbf{m}}_2 \cdot \mathbf{B}_1 \approx \frac{\mu_0 N^2 \pi^2 a^4 I}{4\pi}\frac{3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\hat{\mathbf{m}}_2\cdot\hat{\mathbf{r}}) - \mathbf{m}_1\cdot\hat{\mathbf{m}}_2}{r^3} \tag{34}$$

Thus, for two identical, coaxial coils separately by distance r, their mutual inductance is,

$$M_\parallel \approx \frac{\mu_0 N^2 \pi a^4}{2r^3} \qquad (\mathbf{m}_1 = \mathbf{m}_2 \parallel \mathbf{r}), \tag{35}$$

while for two identical, parallel coils with axes perpendicular to their line of centers their mutual inductance is,

$$M_\perp \approx \frac{\mu_0 N^2 \pi a^4}{4r^3} \qquad (\mathbf{m}_1 = \mathbf{m}_2 \perp \mathbf{r}). \tag{36}$$

An impressive collection of lore on computation of inductances of solenoidal coils is available in [25].

There is a capacitance between adjacent turns of wire in the coils, with value roughly 0.1 nF/m for wire with insulation of relative permittivity $\epsilon = 3$ and insulation of thickness

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and the (time-average) radiated power is,

$$\begin{aligned} P_{\rm rad} &= 2\pi \int \frac{dP_{\rm rad}}{d\Omega}\, d\cos\theta = \frac{1}{2}\frac{8Z_0\pi^5 N^2 a^4}{3\lambda^4}\left[|I_p|^2 + |I_s|^2 + \frac{3}{2}\int d\cos\theta\; Re(I_P^\star I_s\, e^{-ikd\cos\theta})\, \sin^2\theta\right] \\ &= \frac{R_{\rm rad}}{2}\left[|I_p|^2 + |I_s|^2 + \frac{3}{2}\int d\cos\theta\; Re(I_P^\star I_s\, e^{-ikd\cos\theta})\,(1-\cos^2\theta)\right]. \end{aligned} \tag{31}$$

The third (interference) term of eq. (31) is outside the scope of the circuit analysis.

For a more accurate analysis, a numerical simulation with a code such as NEC4 [18] should be performed.

equal to the radius of the wire.[11] For a single-layer solenoid coil of N turns of radius a m, this self capacitance is,

$$C_{\rm self} \approx 2\pi N a \times 10^{-10} \text{ F}. \tag{37}$$

The resonant angular frequency of such a planar spiral coil is,

$$\omega_0 = \frac{1}{\sqrt{LC_{\rm self}}} \approx \frac{1}{\sqrt{8\pi^3 N^3 a^3 \times 10^{-17}/b}} \approx \frac{10^8}{2N\sqrt{2\pi Na/b)a}}\,. \tag{38}$$

For example, a solenoid coil of radius $a = 5$ cm, length $b = 50$ cm and $N = 60$ turns has $\omega_0 \approx 10^7$.

A.2 Planar Spiral Coils

A coil geometry that is becoming popular in wireless transmission of power is a planar spiral, which in its simplest version has N turns with radial spacing r_0 (= diameter of the wire + insulation) for radii $0 < r < a$.

We recall that the self inductance of a circular coil of radius a made from a conductor of radius r_0 can be estimated by suppose that the magnetic field at distance r from the coil is $B \approx \mu_0 I/2\pi r$, and that the flux linked by the coil is $\Phi = \int_{r_0}^a 2\pi a\, dr\, B = \mu_0 I \ln(a/r_0)$, *i.e.*, $L \approx a\ln(a/r_0)$. Applying this argument to a spiral coil, the flux linked is sum/integral of that linked by the $N = a/r_0$ coils of radius r due to the field generated in the N coils labeled by radius r',

$$\begin{aligned}
\Phi &\approx \mu_0 I \int_{r_0}^a \frac{dr}{r_0} r \left[\int_{r_0}^{r-r_0} \frac{dr'}{r_0}\left(\int_{r_0}^{r'} \frac{dr''}{r''} - \int_{r_0}^{r-r'} \frac{dr''}{r''}\right) + \int_{r+r_0}^a \frac{dr'}{r_0}\int_{r'-r}^{r'} \frac{dr''}{r''}\right] \\
&= \frac{\mu_0 I}{r_0^2}\int_{r_0}^a r\, dr \left[\int_{r_0}^{r-r_0} dr'[\ln r' - \ln(r-r')] + \int_{r+r_0}^a dr'[\ln r' - \ln(r'-r)]\right] \\
&= \frac{\mu_0 I}{r_0^2}\int_{r_0}^a r\, dr \left[\int_{r_0}^{r-r_0} dr'\, \ln r' - \int_{r_0}^{r-r_0} ds\, \ln s + \int_{r+r_0}^a dr'\, \ln r' - \int_{r_0}^{a-r} ds'\, \ln s'\right] \\
&= \frac{\mu_0 I}{r_0^2}\int_{r_0}^a r\, dr\, [a\ln a - a - (r+r_0)\ln(r+r_0) - r - r_0 - (a-r)\ln(a-r) + a - r + r_0\ln r_0 - r_0] \\
&\approx \frac{\mu_0 I}{r_0^2}\int_{r_0}^a r\, dr\, (a\ln a + r_0\ln r_0 - 2r_0 - 2r - a\ln r) \\
&\approx \frac{\mu_0 I}{r_0^2}\left[\frac{a^2 - r_0^2}{2}(a\ln a + r_0\ln r_0 - 2r_0) - 2\frac{a^3 - r_0^3}{3} - \frac{a^3\ln a}{2} + \frac{a^3}{2} + \frac{ar_0^2\ln r_0}{2} - \frac{ar_0^2}{2}\right] \\
&\approx -\frac{\mu_0 I}{r_0^2}\frac{a^3}{6} = -\frac{\mu_0 N^2 I a}{6}\,.
\end{aligned} \tag{39}$$

Then, the self inductance is,

$$L = \frac{|\Phi|}{I} \approx \frac{\mu_0 N^2 a}{6}\,. \tag{40}$$

[11] For a discussion of the capacitance of two-wire transmission lines see, for example, [26].

An extensive study [27] of planar spiral coils indicates that the self inductance is closer to $\mu_0 N^2 a/4$.

To estimate the mutual inductance of a coaxial pair of such planar spiral coils, separated by distance r, we note that the magnetic moment of each coil has magnitude,

$$m \approx \int_0^a \frac{dr}{r_0} I\pi r^2 = \frac{\pi I a^3}{3r_0} = \frac{\pi I N a^2}{3}\,, \tag{41}$$

which differs from that of a solenoid by a factor 1/3. Hence, recalling eqs. (34)-(35), the mutual inductance is,

$$M_{\parallel} \approx \frac{\mu_0 N^2 \pi a^4}{6r^3} \approx \frac{a^3}{r^3} L. \qquad (\mathbf{m}_1 = \mathbf{m}_2 \parallel \mathbf{r}). \tag{42}$$

The total length of the conductor of the coil is,

$$l \approx \int_0^a 2\pi r\, \frac{dr}{r_0} = \frac{\pi a^2}{r_0} = \pi N a, \tag{43}$$

so the self capacitance is,

$$C_{\rm self} \approx \pi N a \times 10^{-10}\ \mathrm{F}. \tag{44}$$

The resonant angular frequency of such a planar spiral coil is,

$$\omega_0 = \frac{1}{\sqrt{LC_{\rm self}}} \approx \frac{1}{\sqrt{4\pi^3 N^3 a^2 \times 10^{-17}/4}} \approx \frac{10^8}{N\sqrt{\pi N a}}\,. \tag{45}$$

In this model,

$$(\omega_0 M)^2 \approx \left(\frac{10^8}{N\sqrt{\pi N a}} \frac{\mu_0 N^2 \pi a^4}{6r^3} \right)^2 = \left(\frac{40\pi^2 a^3}{6r^3} \sqrt{\frac{N}{\pi}} \right)^2 \approx 1400 N \frac{a^6}{r^6}\,. \tag{46}$$

If, say, we desire that $(\omega_0 M)^2 = RR_p = 10\ \Omega^2$ when $r = 3$ m for planar spiral coils with radius $a = 0.3$ m, then we need $N \approx 700$ turns (such that $r_0 \approx 0.4$ mm = diameter of wire + insulation). In this case, $\omega_0 \approx 10$ kHz, and $f = 1600$ Hz.

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