A Naïve Estimate of the Coupling Constant in Yukawa Theory

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1 Problem

In 1935, Yukawa [1] postulated that the static force which holds nucleons together in nuclei can be related to a scalar potential, $\phi_g = g\, e^{-\mu r}/r$, where g is a "nuclear charge".[1]

Suppose we attribute all of the proton's rest energy, $m_p c^2$, to the energy of its nuclear force field,[2] where c is the speed of light in vacuum and m_p is the rest mass of the proton. What would this mass be if the proton were a spherical shell of radius a_p?

Hint: $U = \int \rho\phi \, d\mathrm{Vol}/2$ *still holds, where* ρ = *density of nuclear charge.*

Relate ρ *to* ϕ *via an appropriate generalization of Poisson's equation,* $\nabla^2\phi = -4\pi\rho$*, where* ρ *is the volume density of charge. You should find* $U = \int [(\boldsymbol{\nabla}\phi)^2 + \mu^2\phi^2]\, d\mathrm{Vol}/8\pi$.

In quantum theory, the coupling constants $e^2/\hbar c = \alpha \approx 1/137$ and $g^2/\hbar c$ play important roles. Given that $m_p/m_e = 1836$, estimate the pion-nucleon coupling constant $g^2/\hbar c$ supposing the proton is a spherical shell of nuclear charge of radius $a_p = 0.86 \times 10^{-13}$ cm, and the electron is a spherical shell of electric charge of radius such that all of the electron's rest mass, m_e, is electromagnetic.

This estimate agrees fairly well with experiment. Is this physics or numerology?

2 Solution

Poisson's equation in electrostatics, for the potential ϕ_e due to a static density ρ_e of electric charge, is,[3]

$$\nabla^2\phi_e = -4\pi\rho_e. \tag{1}$$

Yukawa's equation [1, 2] for the static nuclear potential ϕ_g is $(\nabla^2 - \mu^2)\,\phi_g = 0$, away from a point source of nuclear charge g at the origin, for which,

$$\phi_g = g\frac{e^{-\mu r}}{r}\,, \tag{2}$$

for some constant μ (with dimensions of inverse length). Away from the origin,

$$\nabla^2\phi_g = \frac{1}{r}\frac{\partial}{\partial r^2}(r\phi_g) = \mu^2\phi_g, \tag{3}$$

[1] For a brief introduction to Yukawa theory by the author, see p. 226 of [2].

[2] The mass of the neutron is about 0.14% higher than the mass of the proton, which suggests that the energy of the proton's electromagnetic field does not contribute significantly to its mass.

[3] See, for example, pp. 10-10a of [3].

while close to the origin, $\phi_g \approx g/r$, for which,

$$\nabla^2 \phi_g \approx g \nabla^2 (1/r) = -4\pi g\, \delta^3(\mathbf{r}) \tag{4}$$

recalling that $\nabla^2(1/r) = -4\pi\, \delta^3(\mathbf{r})$.[4]

This suggests that in case of a (static) volume density ρ_g of nuclear charge, the Yukawa potential is,

$$\phi_g(\mathbf{r}) = \int \frac{\rho_g(\mathbf{r}')\, e^{-\mu|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d\mathrm{Vol}', \tag{5}$$

and Poisson's equation becomes,[5]

$$(\nabla^2 - \mu^2)\, \phi_g = -4\pi \rho_g. \tag{6}$$

The concept of the potential is that the interaction energy of two (point) charges g_1 and g_2 is,

$$U_{12} = g_1 \phi_{g,12} = g_2 \phi_{g,21}. \tag{7}$$

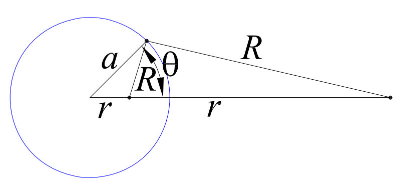
For a collection of particles, this leads to the interaction energy,

$$U = \frac{1}{2} \sum_{i,j} g_i \phi_{g,ij} \to \frac{1}{2} \int \rho_g(\mathbf{r}) \phi_g(\mathbf{r})\, d\mathrm{Vol}. \tag{8}$$

Then, with ρ_g from eq. (6), we have, recalling that $\phi_g \nabla^2 \phi_g = \boldsymbol{\nabla}(\phi_g \boldsymbol{\nabla} \phi_g) - (\boldsymbol{\nabla}\phi_g)^2$, and using Gauss' theorem,

$$\begin{aligned} U = -\frac{1}{8\pi} \int \phi_g (\nabla^2 - \mu^2) \phi_g\, d\mathrm{Vol} &= \frac{1}{8\pi} \int \left[-\boldsymbol{\nabla}(\phi_g \boldsymbol{\nabla}\phi_g) + (\boldsymbol{\nabla}\phi_g)^2 + \mu^2 \phi_g^2 \right] d\mathrm{Vol} \\ &= \frac{1}{8\pi} \int \left[(\boldsymbol{\nabla}\phi_g)^2 + \mu^2 \phi_g^2 \right] d\mathrm{Vol}, \end{aligned} \tag{9}$$

for a charge distribution that is nonzero only within a bounded volume, such that $\phi_g \boldsymbol{\nabla} \phi_g \propto 1/r^3$ for large r.



We now consider a nuclear charge g that is uniformly distributed over a spherical shell (about the origin) of radius a. At a point on the z-axis at distance r from the origin,

$$\phi_g(r) = \int_{-1}^{1} d\cos\theta\, \frac{g}{2} \frac{e^{-\mu R}}{R}\,, \tag{10}$$

[4]See, for example, pp. 39-40 of [4].

[5]Equation (6) is sometimes called the screened Poisson equation.

where,

$$R^2 = a^2 + r^2 - 2ar\cos\theta, \qquad 2R\,dR = -2ar\,d\cos\theta, \tag{11}$$

and when $\cos\theta = \pm 1$, $R = a + r,\ |a - r|$. Hence,

$$\phi_g(r > a) = \frac{g}{2ar}\int_{r-a}^{r+a} dR\, e^{-\mu R} = \frac{g}{2\mu ar}(e^{-\mu(r-a)} - e^{-\mu(r+a)}) = \mu g \frac{\sinh\mu a}{\mu a}\frac{e^{-\mu r}}{\mu r}\,, \tag{12}$$

$$\phi_g(r < a) = \frac{g}{2ar}\int_{a-r}^{a+r} dR\, e^{-\mu R} = \frac{g}{2\mu ar}(e^{-\mu(a-r)} - e^{-\mu(a+r)}) = \mu g \frac{e^{-\mu a}}{\mu a}\frac{\sinh\mu r}{\mu r}\,. \tag{13}$$

For $\mu = 0$ we recover the form for ordinary electrostatics of a spherical shell of electric charge q, $\phi(r > a) = q/r$, while $\phi(r < a) = q/a$.

The gradient of the potential (12)-(13) is purely radial,

$$\boldsymbol{\nabla}\phi_{g,r}(r > a) = \mu\frac{\partial\phi_g(r > a)}{\partial\mu r} = -\frac{\mu g\sinh\mu a}{a}\left(\frac{e^{-\mu r}}{\mu^2 r^2} + \frac{e^{-\mu r}}{\mu r}\right)\,, \tag{14}$$

$$\boldsymbol{\nabla}\phi_{g,r}(r < a) = \mu\frac{\partial\phi_g(r < a)}{\partial\mu r} = -\frac{\mu g\, e^{-\mu a}}{a}\left(\frac{\sinh\mu r}{\mu^2 r^2} - \frac{\cosh\mu r}{\mu r}\right)\,, \tag{15}$$

and the field energy (9) is, noting that $\sinh^2 x = (\cosh 2x - 1)/2$, $2\sinh x\cosh x = \sinh 2x$, and $\cosh^2 x = (\cosh 2x + 1)/2$,

$$\begin{aligned}
U &= \frac{g^2\sinh^2\mu a}{2\mu a^2}\int_{\mu a}^{\infty}(\mu r)^2\, d(\mu r)\, e^{-2\mu r}\left(\frac{1}{\mu^4 r^4} + \frac{2}{\mu^3 r^3} + \frac{2}{\mu^2 r^2}\right) \\
&\quad + \frac{g^2\, e^{-2\mu a}}{2\mu a^2}\int_0^{\mu a}(\mu r)^2\, d(\mu r)\left(\frac{\cosh 2\mu r - 1}{2\mu^4 r^4} - \frac{\sinh 2\mu r}{\mu^3 r^3} + \frac{\cosh 2\mu r}{\mu^2 r^2}\right) \\
&= \frac{g^2}{2\mu a^2}\frac{\cosh 2\mu a - 1}{2}\, e^{-2\mu a}\left(1 + \frac{1}{\mu a}\right) + \frac{g^2\, e^{-2\mu a}}{2\mu a^2}\left(\frac{1 - \cosh 2\mu a}{2\mu a} + \frac{\sinh 2\mu a}{2}\right) \\
&= \frac{g^2\, e^{-2\mu a}}{2\mu a^2}\left(\frac{\cosh 2\mu a - 1 + \sinh 2\mu a}{2}\right) = \frac{g^2(1 - e^{-2\mu a})}{4\mu a^2}\,,
\end{aligned} \tag{16}$$

using Dwight [5] 568.2 and 678.12. If $\mu \to 0$, this goes to $g^2/2a$, as expected from electrostatics.

Applying this model to a proton, its rest mass would be,

$$m_p = \frac{U_p}{c^2} = \frac{g^2(1 - e^{-2\mu a_p})}{4\mu a_p^2 c^2}\,, \tag{17}$$

where a_p is the radius of the proton (taken to be a spherical shell of nuclear charge).

In nuclear interactions, the range of the Yukawa interaction is about the same as the radius of the proton, $a_p \approx 0.86 \times 10^{-13}$ cm,[6] *i.e.*, $\mu a_p \approx 1$, in which case $U_p \approx g^2/4a_p$ for our model of a proton as a spherical shell of nuclear charge.

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[6] https://en.wikipedia.org/wiki/Proton_radius_puzzle

For an electron modeled as a spherical shell of electric charge e of radius a_e, the electric-field energy $U_e = e^2/2a_e$ equals the electron rest mass (times c^2) for $a_e = r_e/2 = 1.4 \times 10^{-13}$ cm, where $r_e = m_e c^2/e^2$ is the so-called classical election radius.[7]

Experimentally, $m_p/m_e = 1836$, so our models imply that,

$$\frac{m_p}{m_e} \approx \frac{g^2/4a_p}{e^2/2a_e}, \qquad \frac{g^2}{\hbar c} \approx \frac{e^2}{\hbar c}\frac{m_p}{m_e}\frac{2a_p}{a_e} \approx \frac{1}{137} \cdot 1836 \cdot 2 \cdot \frac{0.86}{1.4} \approx 16.5. \tag{18}$$

This compares fairly well with the experimental value of ≈ 14.2.[8]

However, the above discussion was for a repulsive Yukawa potential, which doesn't explain why nucleons stick together. If we change the sign in eqs. (2) and (5) to have an attractive potential, then the field energy (9) would be negative, which seems unphysical. Hence, the model presented in this problem cannot be taken too seriously.

References

[1] H. Yukawa, *On the Interaction of Elementary Particles. I*, Proc. Phys.-Math. Soc. Japan **17**, 48 (1935), http://kirkmcd.princeton.edu/examples/EP/yukawa_ppmsj_17_48_35.pdf

[2] K.T. McDonald, *Ph501 Lecture 19*, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture19.pdf

[3] K.T. McDonald, *Ph501 Lecture 1*, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture1.pdf

[4] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, 1975), http://kirkmcd.princeton.edu/examples/EM/jackson_ce2_75.pdf

[5] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[7] https://en.wikipedia.org/wiki/Classical_electron_radius

[8] See, for example, https://arxiv.org/pdf/hep-ph/0009312.pdf, where our coupling constant $g^2/\hbar c$ is their $g^2/4\pi$.