(1) Excellent — I have no important changes to suggest

A SOLUTION TO THE PROBLEM OF APOLLONIUS

by

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for

Mr. Wilson

English 8-2

ABSTRACT

One of the most famous problems of classical geometry is the ruler and compass construction of a circle tangent to three given circles. This paper demonstrates one solution, accredited to Apollonius, the Greek mathematician whose name the problem bears. Although the value of a solution to, say, draftsmen is not small, the primary interest in it derives from the mathematical beauty of the problem and the elegance of its solution. The particular solution presented here is also significant for the insight it offers into the nature of geometric proofs. Reduction of the problem to simpler terms, always an ideal, proves especially useful when applied to the problem of Apollonius. Because circles and points have a simple geometric relation, the problem of constructing a circle tangent to three others reduces easily to the case of finding a circle tangent to only one circle, but also passing through two given points. Of course, the reduction is more complicated than merely replacing a circle by the point at its center, but the solution to the problem of Apollonius does follow from straightforward application of elementary geometric principles.

LIST OF FIGURES

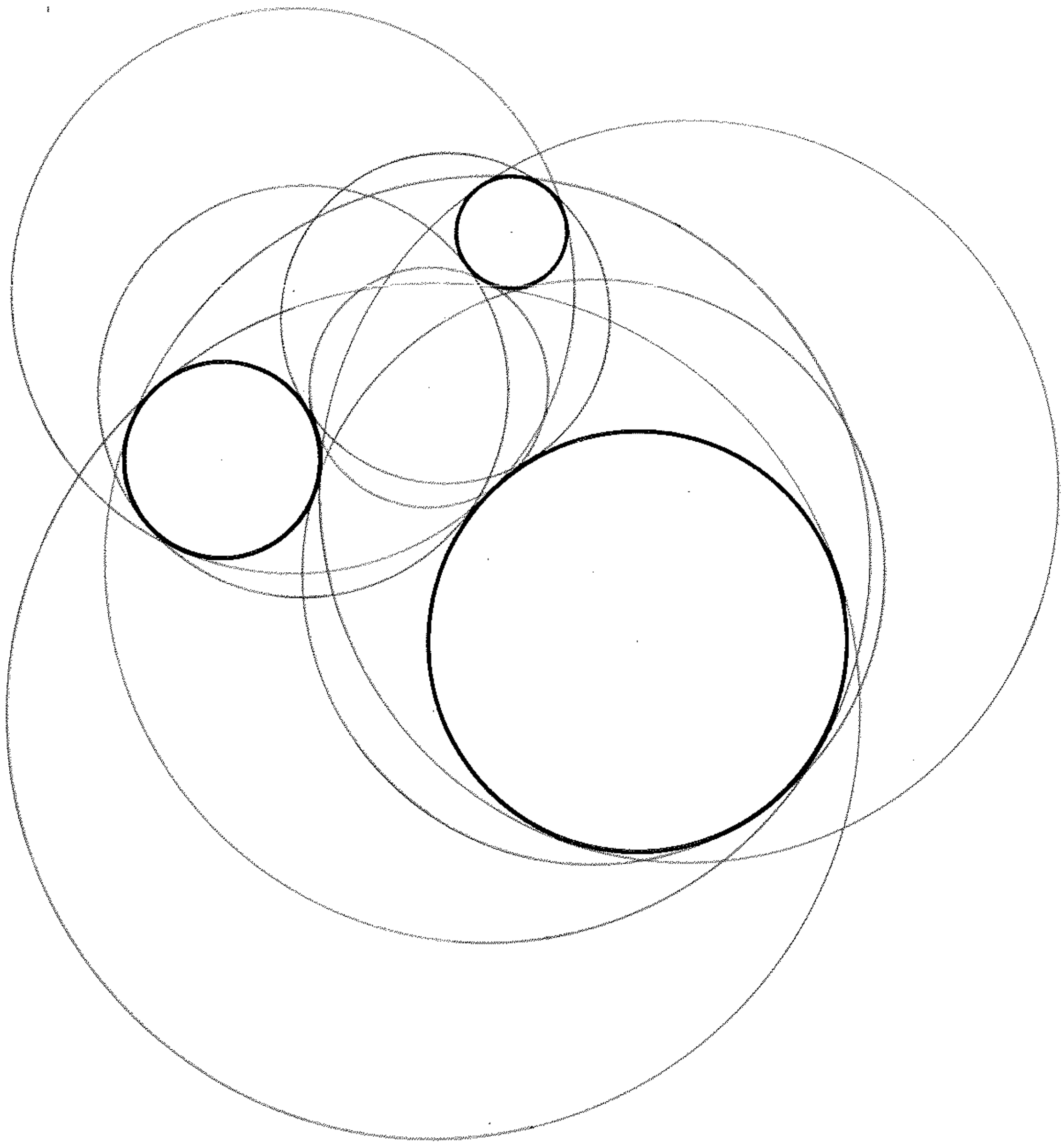


Figure 1. The Eight Solutions to the Problem of Apollonius.

A SOLUTION TO THE PROBLEM OF APOLLONIUS

INTRODUCTION

It is ironic that the science of geometry is better known for its inability to solve certain elementary problems, trisecting an angle, squaring the circle, than for its power to solve relatively difficult ones. A noteworthy success of geometry has come in the solution to the problem of Apollonius, that of constructing a circle tangent to three given circles using only a compass and straight-edge. Since the time the problem was first put forth by Apollonius about 2200 years ago, it has continually fascinated geometers with the result that approximatley 70 different solutions are known today. The particular solution demonstrated here derives from Apollonius himself, and correspondingly, the approach to the problem displays a classical simplicity. Because the problem of constructing a circle tangent to three others does not yield immediately to elementary geometric analysis, it is reduced to the simpler problem of constructing a circle tangent to two circles and passing through a fixed point. This simplified problem is still too difficult for direct solution, so it too is reduced, now to the case of finding a circle tangent to one circle and passing through two fixed points. Working backwards, solving the simplest problem first and building upon the result, the complete solution to the problem of Apollonius follows at once. When the three given circles are mutually external, the only case considered here, there are eight different circles, shown red in Figure 1, all tangent to the given three.

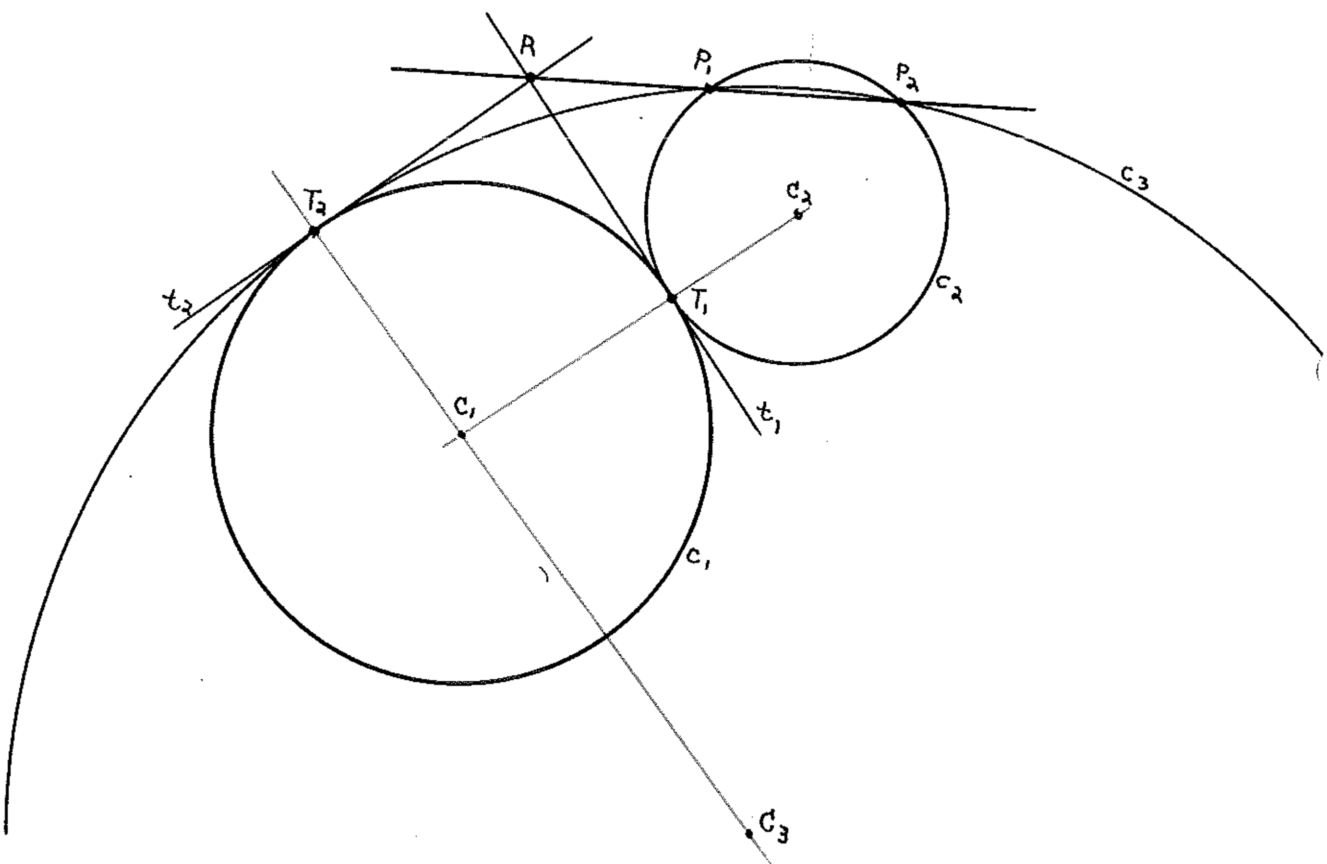


Figure 2. Solution to the One Circle-Two Point Problem.

II. DEVELOPMENT OF THE SOLUTION

A. One Circle-Two Point Problem

As a preliminary to the complete solution, a construction for a circle tangent to one circle and passing through two fixed points must be found. To do this, some concepts beyond high school plane geometry are necessary and these will be developed when required. The proofs of theorems and constructions normally found in a high school geometry course will be assumed in order to shorten the discussion. The material presented in this paper appears in extended form scattered over 150 pages of College Geometry by Daus (1). Our work has been not so much original as it has been condensation of known proofs, reexpressing them in a more unified order and in slightly more elementary terms. Two other references which proved helpful in the subsequent derivations were College Geometry by Altshiller-Court (2), and Modern College Geometry by Davis (3).

1. Some Observations. Turning to the one circle-two point problem, let us first consider the problem already solved and therefrom determine various properties of the solution. Figure 2 shows a circle tangent to another and passing through two points. In fact, we see that if c_1 is the given circle (captial C stands for the center of a circle and small c for the circle itself), and P_1 and P_2 are the fixed points, then there are two different circles, c_2 and c_3, which satisfy the problem. Of course, if c_1 separates P_1 from P_2, then there exists no solution, but the criterion for the problem of Apollonius that the three given circles are mutually external prohibits this unusual case from occurring.

Examining Figure 2, we notice that the common tangent of circles c_1

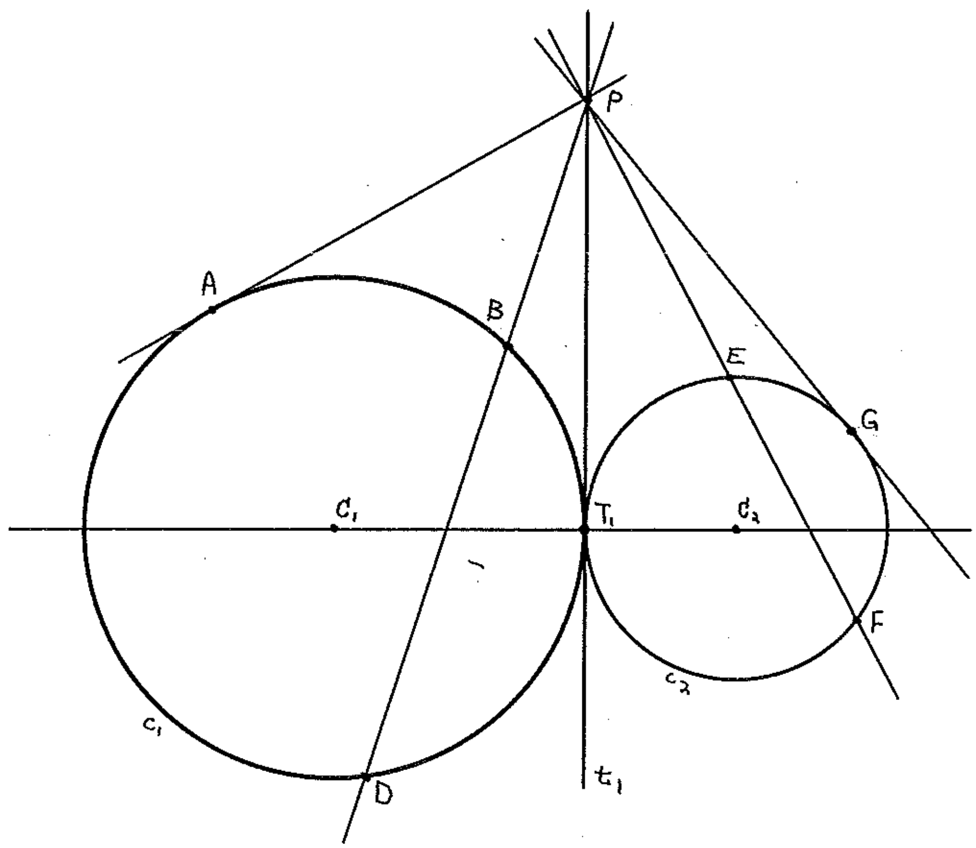


Figure 3. The Radical Axis of Tangent Circles.

and c_2, indicated by line t_1, seems to meet line t_2, the common tangent of circles c_1 and c_3, in a point R which is on line P_1P_2. This is our clue to the solution, and detailed investigation of point R follows. To prove that the three lines mentioned actually do meet in a point, some properties of these lines must be determined. For ease of discussion, Figure 2 has been partially redrawn in Figure 3 to emphasize line t_1. The line of centers of circles c_1 and c_2 cuts the circles at T_1, the point of tangency, and the common tangent t_1 is perpendicular to line C_1C_2 at T_1. From elementary geometry we see that the tangents PA, PT_1, and PG from any point P on the common tangent are equal. Recall that a tangent to a circle from a point is the mean proportional between a secant to the circle from the same point and the external segment of that secant. Thus, since PBD and PEF are secants,

$$PA^2 = PB \times PD = PT_1^{\,2} = PE \times PF = PG^2.$$

2. Power and the Radical Axis. For convenience we define the power of a point with respect to a circle as the product of a secant to the circle from that point and the external segment of the secant. In Figure 3, line t_1 is the locus of points which have equal powers with respect to circles c_1 and c_2 as shown by the equation above and is called the radical axis of the circles. Similarily, in Figure 2, line t_2 is the radical axis of circles c_1 and c_3, and line P_1P_2 is the radical axis circles c_2 and c_3. We note here that the radical axis of any two intersecting circles is their common chord extended(for further definitions of the radical axis and other terms, see the GLOSSARY). Again in Figure 2, line P_1P_2 meets line t_2 in a point whose power with respect to all three circles c_1, c_2, and c_3 is equal. Likewise line P_1P_2 intersects t_1 in a point with equal powers to all three circles. If these two points of intersection are not the same, a contradiction occurs, and therefore,

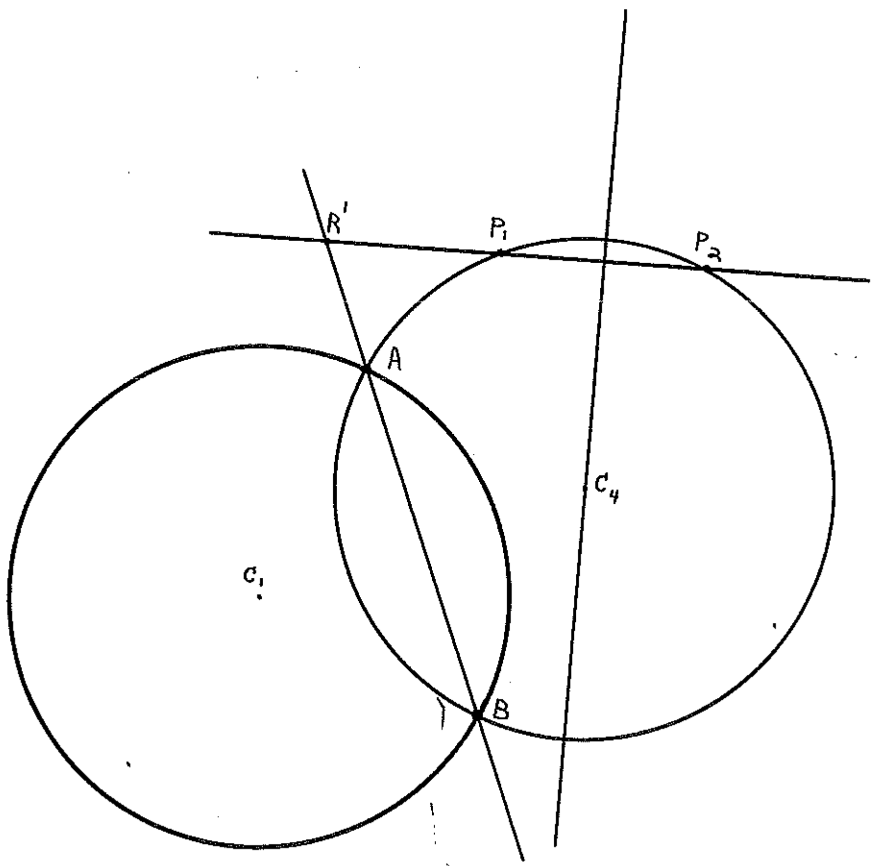


Figure 4. Construction for the Radical Center.

the three lines intersect in a unique point, called the radical center of the three circles.

Now that we know that lines P_1P_2, t_1, and t_2 intersect in a point R, a means of finding circles c_2 and c_3 from points P_1 and P_2 and circle c_1 appears. When by some method point R has been determined, the points of tangency T_1 and T_2 can be easily constructed. Then line C_1T_1 is the line of centers of circles c_1 and c_2, and line C_1T_2 is the line of centers of c_1 and c_3. The centers of circles c_2 and c_3 will then be the intersections of lines C_1T_1 and C_1T_2 with the perpendicular bisector of line P_1P_2.

3. Finding the Radical Center. Once the radical center, R, of circles c_1, c_2, and c_3 has been found, the solution to the one circle-two point problem will be complete. Towards finding R, we notice that line P_1P_2, which contains R, is the radical axis of any two circle passing through P_1 and P_2, as well as of c_2 and c_3. Suppose a circle c_4 passes through P_1 and P_2 and also intersects c_1 at A and B as shown in Figure 4. Then the radical axis of c_1 and c_4 is line AB, which intersects P_1P_2 at R'. The point R' has equal powers with respect to c_1 and c_4 and also that same power with respect to c_2 and c_3, since line P_1P_2R' is the radical axis of all three circles c_2, c_3, and c_4. Therefore R' is R, the radical center of c_1 and the desired circles, c_2 and c_3 (points R and R' of Figures 2 and 4 do superimpose).

4. The Solution. The rather lengthy discussion above can now be condensed into a few steps which will provide the solution to the one circle-two point problem. First, construct a circle passing through the given points, P_1 and P_2, and intersecting the given circle. The center of this circle will, of course, be on the perpendicular bisector of line segment P_1P_2. Next, find the intersection of P_1P_2 with the common chord

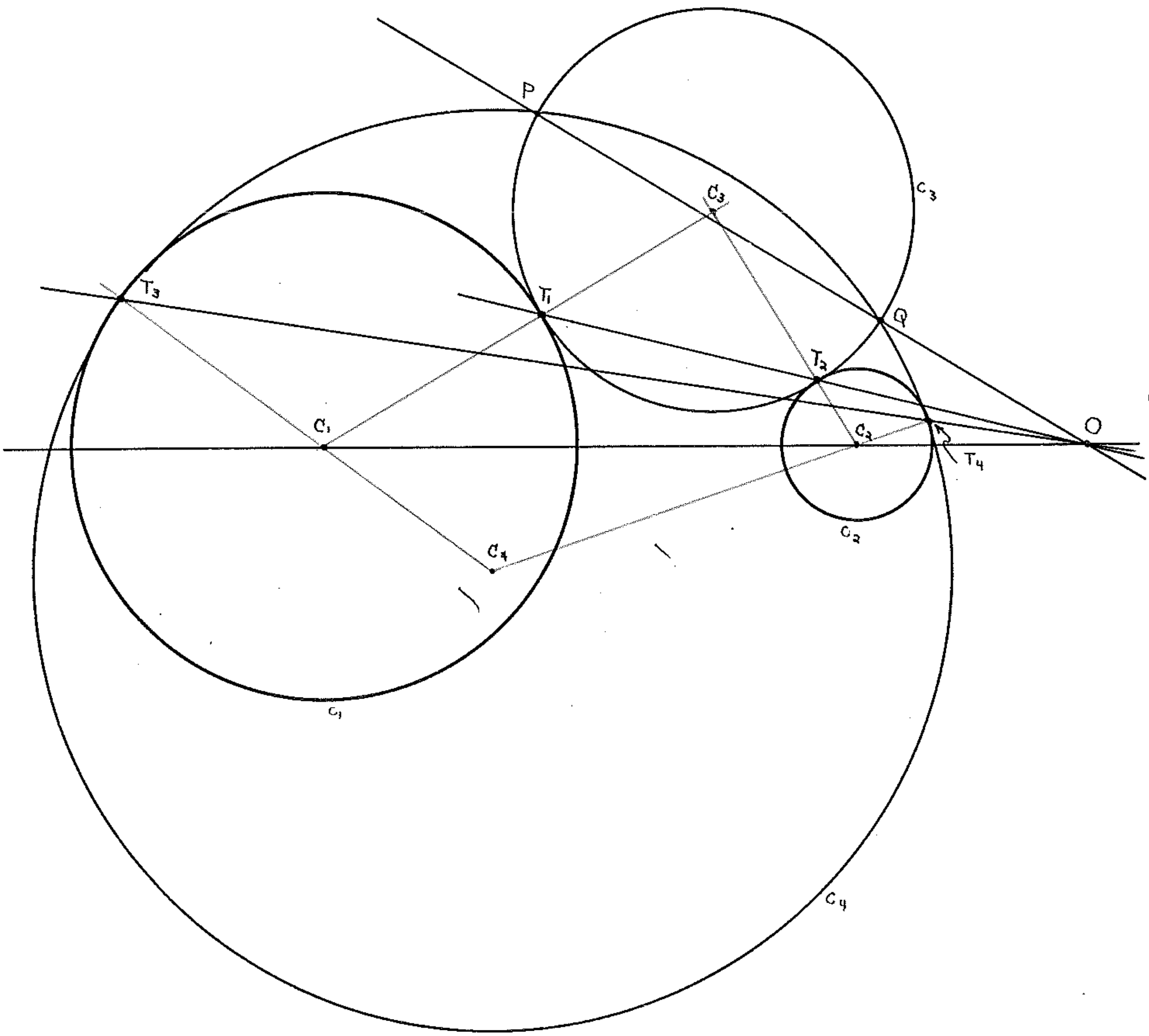


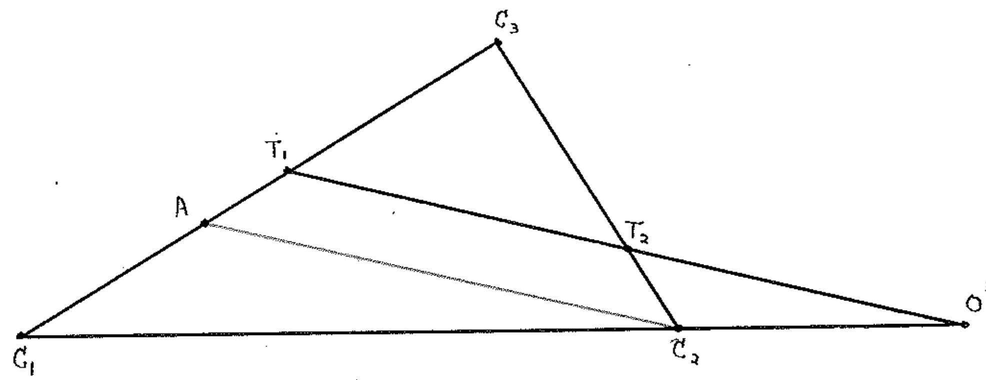
Figure 5. Solution to the Two Circle-One Point Problem.

of the two circles (as in Figure 4). From R, the point of intersection, construct tangents to the given circle. The lines from the center of the given circle through the two points of tangency then meet the perpendicular bisector of line P_1P_2 at the centers of the two circles tangent to the given circle and passing through P_1 and P_2, shown in Figure 2. These steps constitute the solution to the problem of Apollonius with two of the three circles reduced to points.

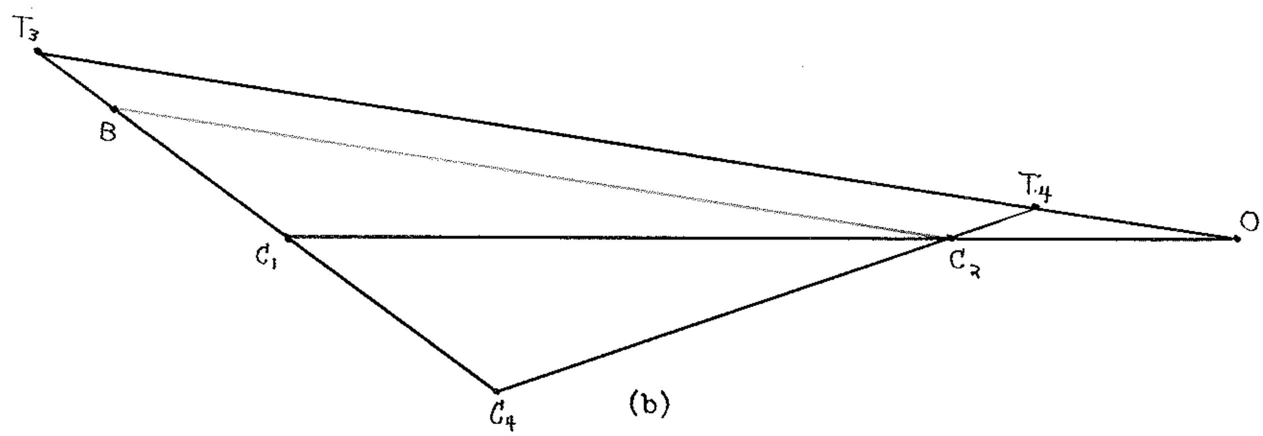
B. Two Circle-One Point Problem.

1. Examining the Problem. When only one of the three given circles is reduced to a point, the problem becomes that of construction a circle tangent to two given circles and passing through one point. Again using the device of considering the solved problem as a means of developing a construction, the solution appears in Figure 5. Circles c_1 and c_2 are given along with point P. Tangent to both c_1 and c_2 are the circles c_3 and c_4, which also pass through P. Unless the special case arises where c_3 and c_4 are tangent at P, they intersect each other at a point Q as well as P. If the point Q were somehow known, then circles passing through P and Q, and tangent to either c_1 or c_2, would satisfy the conditions of the two circle-one point problem. Once Q has been found, the problem reduces to the one circle-two point case, whose solution is now known.

2. Quantitative Observations. In Figure 5, the points of tangency of the various circles are T_1, T_2, T_3, and T_4, and we notice that lines PQ, T_1T_2, and T_3T_4 all appear to meet line C_1C_2 at the point O. This is the lead which shortly provides the solution. To prove that lines T_1T_2 and T_3T_4 do intersect on line C_1C_2, we must have some quantitative information about these lines. The red lines in Figure 5 connect the points of



(a)



(b)

Figure 6. Triangles Involving Points of Tangency.

tangency with the centers of the circles, using the theorem from elementary geometry that if two circles are tangent, the line of centers of the circles passes through the point of tangency. For convenience, the pertinent lines appear in Figure 6; those involving circles c_1, c_2, and c_3 in 6a, and those involving c_1, c_2, and c_4 in 6b. Considering first Figure 6a, we see that triangle $C_3T_1T_2$ is isosceles as $C_3T_1 = C_3T_2 = r_3$, the radius of circle c_3. Likewise C_1T_1 equals r_1, radius of c_1, and C_2T_2 equals r_2, radius of c_2. Line T_1T_2 meets line C_1C_2 at O', the point under investigation. If line C_2A is drawn parallel to T_1T_2O', then $T_1A = T_2C_2 = r_2$ because $T_1T_2C_2A$ is an isosceles trapezoid. By the law of similar triangles, here applied to triangles C_1C_2A and $C_1O'T_1$, we have $C_1T_1/AT_1 = O'C_1/O'C_2$, or substituting for C_1T_1 and AT_1,

$$r_1/r_2 = O'C_1/O'C_2 = (O'C_2 + C_1C_2)/O'C_2 = 1 + C_1C_2/O'C_2.$$

Similarliy in Figure 6b, since BC_2 is parallel to T_3T_4O'',

$$r_1/r_2 = O''C_1/O''C_2 = (O''C_2 + C_1C_2)/O''C_2 = 1 + C_1C_2/O''C_2.$$

Equating these two expressions for r_1/r_2, we have

$$1 + C_1C_2/O'C_2 = 1 + C_1C_2/O''C_2 \quad \text{or} \quad O'C_2 = O''C_2.$$

Thus points O' and O'' coincide, and lines T_1T_2 and T_3T_4 meet C_1C_2 extended at a point O such that $OC_1/CC_2 = r_1/r_2$.

3. The Homothetic Center. The point O which divides the line of centers of any two circles in the ratio of their radii is called the homothetic center of those circles. Actually two points will satisfy this condition, one between the centers of the circles, the internal homothetic center, and another, named the external homothetic center, outside the centers, but, of course, on the line of centers. The point O of Figure 5 is the external homothetic center of circles c_1 and c_2,

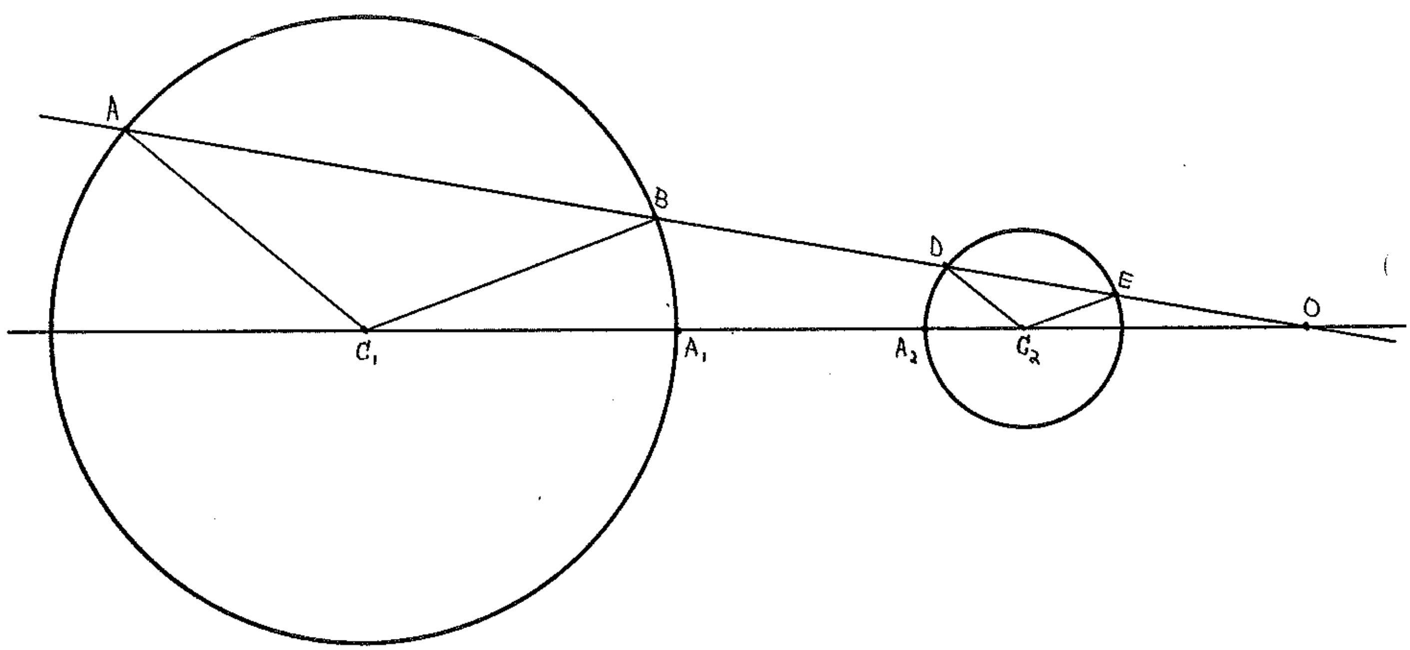


Figure 7. The Homothetic Center.

because $OC_1/OC_2 = r_1/r_2$. We have seen how the lines joining the points of tangency of circles c_3 and c_4 to c_1 and c_2 pass through the external homothetic center of c_1 and c_2. We can ask, are there circles passing through P tangent to c_1 and c_2 such that the lines containing the points of tangency pass through the internal homothetic center? The answer is that two such circles exist, the only other solutions to the two circle-one point problem, but their nature will not concern us here.

4. Homothetic Center and Radical Axis. Before we can prove that line PQ of Figure 5 passes through O, some more properties of the homothetic center must be developed. In fact, since PQ is the radical axis of circles c_3 and c_4, if O is on PQ, then $OT_1 \times OT_2$ must be equal to $OT_3 \times OT_4$. That is, the powers of O with respect to c_3 and c_4 must be equal for O to be on the radical axis. Figure 7 reproduces that part of Figure 5 which concerns proving O is on PQ. Line ABDEO is any line from O which cuts circles c_1 and c_2. Recalling that $OC_1/OC_2 = r_1/r_2$, and noticing that $C_1B = r_1$ and $C_2E = r_2$, we have triangle OC_1B similar to triangle OC_2E. Therefore, $OB/OE = r_1/r_2$. Likewise triangle OC_1A is similar to OC_2D, and $OA/OD = r_1/r_2$. The property of the homothetic center that angle OC_1B equals angle OC_2E provides a construction for O given only circles c_1 and c_2. As a corallary, the line tangent externally to c_1 and c_2 also passes through O.

Returning to the calculations, we see that the relation $OB/OE = OA/OD = r_1/r_2$ holds for any line through O which cuts c_1 and c_2. Therefore, it is useful to write $OB/OE = OA/OD = r$, where r is a constant depending only on circles c_1 and c_2. By the theorem involving secants mentioned above, $OD \times OE = k$, a constant, for all lines OED. (NO NEW PARAGRAPH)

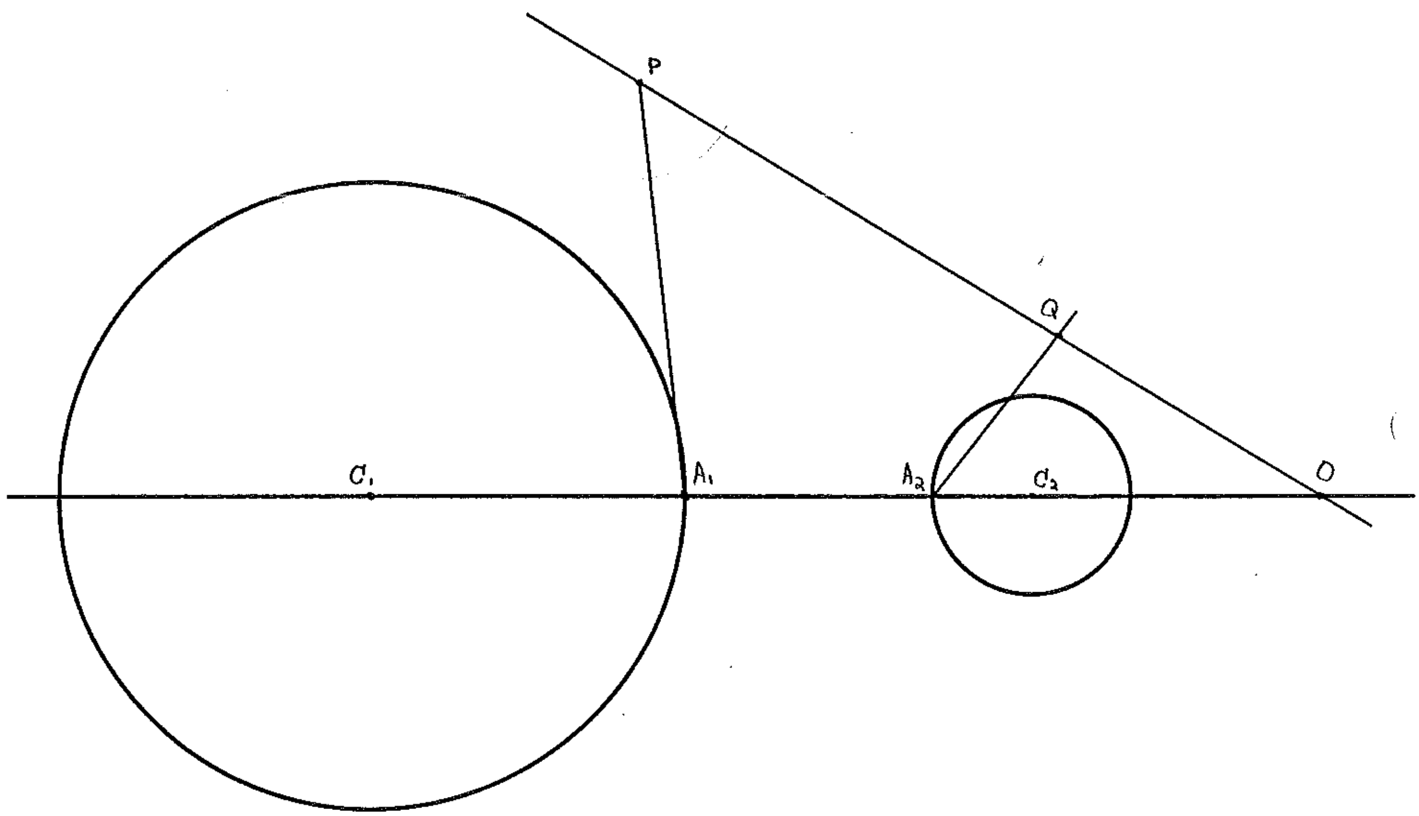


Figure 8. Construction for Point Q.

We can now write $OE = k/OD$ and $OE = OB/r$. Therefore $k/OD = OB/r$, or $OB \times OD = kr$. From the equation $OB/OE = OA/OD$, we get $OB \times OD = OA \times OE = kr$ for all lines OA. For some position of OA, points B and D coincide with T_1 and T_2 of Figure 5. Therefore, $OT_1 \times OT_2 = kr$. At some other position of OA, points A and E coincide with T_3 and T_4, so that $OT_3 \times OT_4 = kr$. Thus $OT_1 \times OT_2 = OT_3 \times OT_4$, which proves that O is on PQ, the radical axis of circles c_3 and c_4.

5. Reduction to the One Circle-Two Point Problem. We are now very close to the solution of the two circle-one point problem, which means determining point Q, and using the one circle-two point construction to find c_3 and c_4. In Figure 7, $OB \times OD = kr$ for all lines OBD, including the line of centers $OC_2A_2A_1C_1$, where A_1 and A_2 are the intersections of circles c_1 and c_2 with line OC_2C_1. Then $OA_1 \times OA_2 = kr = OT_1 \times OT_2$. Since OPQ and OT_2T_1 are secants from O to c_3, $OT_1 \times OT_2 = OP \times OQ$, and therefore $OP \times OQ = OA_1 \times OA_2$. Figure 8 demonstrates the usefulness of this result. Triangle OA_1P is similar to triangle OQA_2 because they share a common vertex angle and have sides in proportion, given by $OA_1/OP = OQ/OA_2$. Therefore angle OPA_1 equals angle OA_2Q. This property determines the point Q given only c_1 and c_2 and P. With Q thus known, application of the one circle-two point construction immediately yields circles c_3 and c_4.

6. A Construction. After all the effort which has gone into the proof of the solution to the two circle-one point problem, the construction can be summarized very briefly. First find the external homothetic center of the given circles c_1 and c_2 by constructing parallel radii and connecting the end points. The intersection of this line with the line of centers C_1C_2 determines the external homothetic center O as in Figure 7. Next draw lines OP and A_1P as in Figure 8. Construct angle

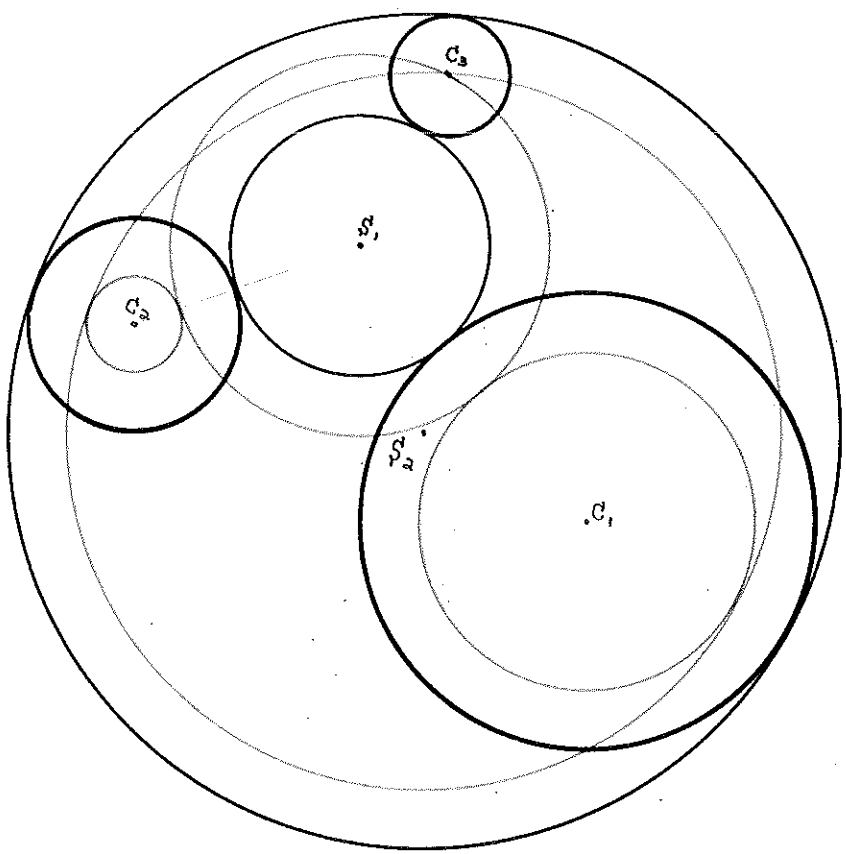


Figure 9. Solutions 1 and 2, r_3 Subtracted.

OA_2Q equal to angle OPA_1, fixing point Q on line OP. Then use the one circle-two point construction with points P and Q and either circle c_1 or c_2 to find the two circles tangent to c_1 and c_2, and passing through P. Again, Figure 5 shows the complete solution to the two circle-one point problem. The above method works even when circle c_1 intersects c_2, a case which may arise in the course of the solution to the problem of Apollonius.

C. Three Circle Problem.

1. Four Pairs of Solutions. We now have enough information and techniques of construction to attack the problem of Apollonius itself. Again we shall assume that the three given circles are mutually external. All eight possible solutions appear in Figure 1, but they cannot be analyzed easily with so many in one drawing. Inspection of the figure, however, reveals that the eight solution naturally group into four pairs, shown in Figures 9 through 12. Notice how the two solutions of each pair are the converses of one another. For example, in Figure 10, black circle s_3 contains only c_2, while s_4 contains c_1 and c_3. A similar statement holds for the other three pairs of solutions.

2. Reduction of c_1. The three circle problem consists of examining the solutions to the problem of Apollonius to find a way in which one circle may be reduced to a point, so that the known solution to the two circle-one point problem can be applied. Consider first Figure 9. If the radius of circle c_3 is subtracted from each of the three given circles c_1, c_2 and c_3, circle c_3 becomes a point while the circles with centers at C_1 and C_2 become the red circles with appropriate radii.

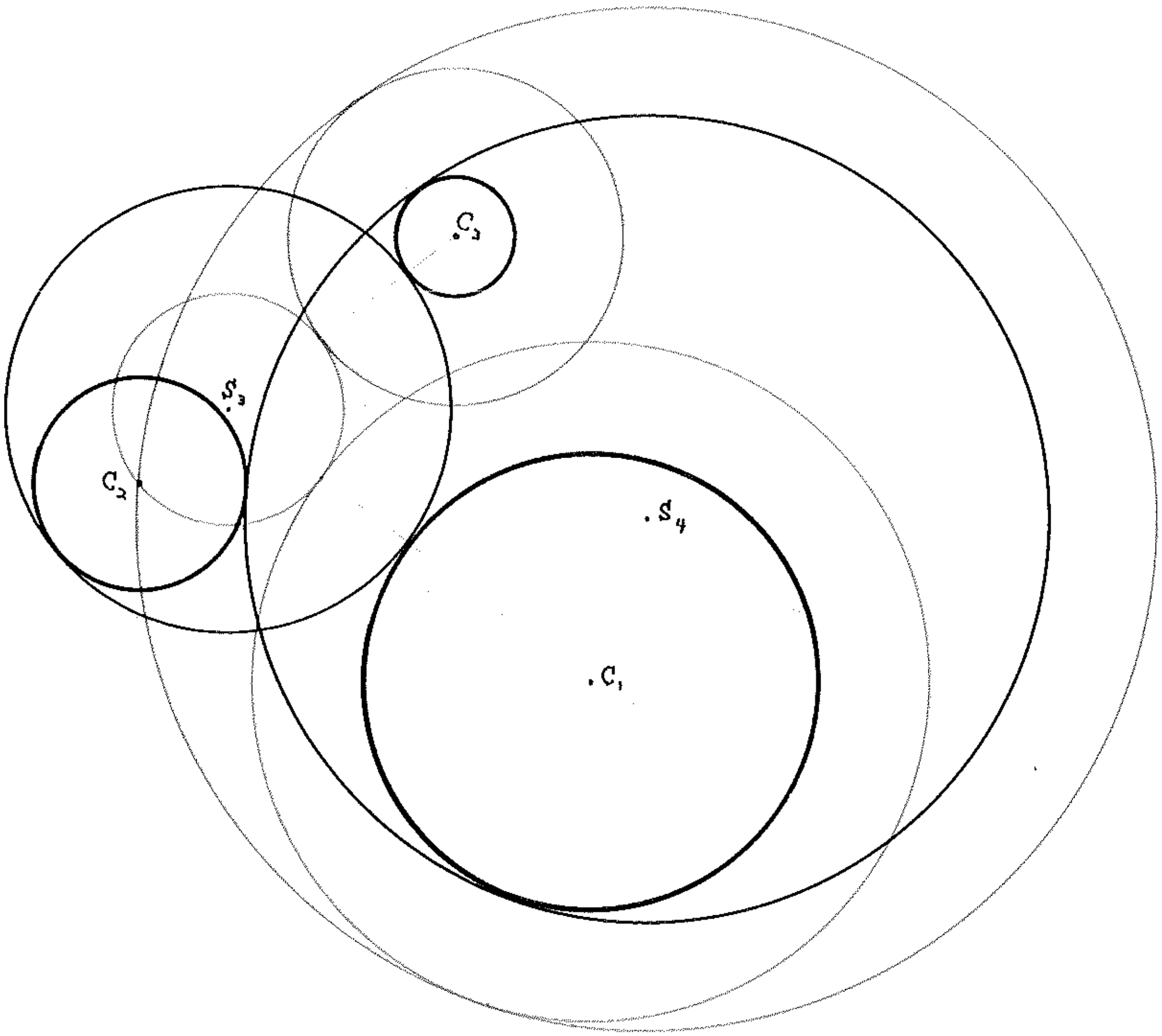


Figure 10. Solutions 3 and 4, r_2 Added.

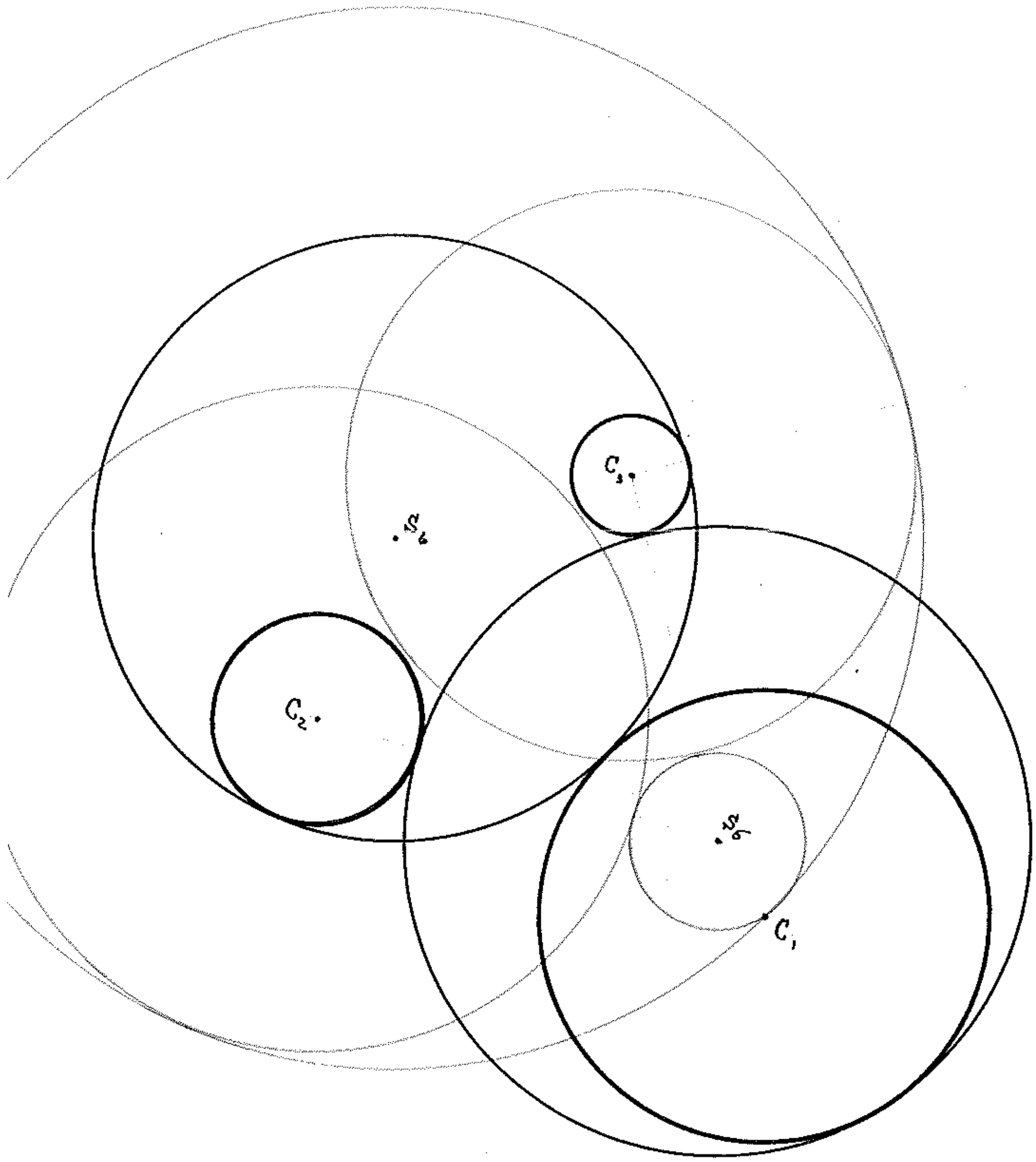


Figure 11. Solutions 5 and 6, r_1 Added.

The two red circles which are tangent to the reduced circles c_1 and c_2 and passing through C_3 have their centers at points S_1 and S_2, the centers of the black solution circles. The radius of red circle s_1 is greater than that of the black s_1 by an amount equal to the radius of c_3. The red s_2 is, however, less than black s_2 by the radius of c_3, which occurs because s_2 is the converse of s_1. A solution to the three circle problem follows immediately from the above method of reduction. To find solutions s_1 and s_2 simply subtract the radius of c_3 from c_1 and c_2, and then find the centers of the two circles tangent to the reduced circles c_1 and c_2 and passing through C_3, by the two circle-one point construction. The centers of these two circles are also the centers of solutions s_1 and s_2.

3. Reduction of c_2. In an effort to generalize the above solution to the three circle problem, we might surmise that in Figure 10, say, subtraction of the radius of c_2 from circles c_1 and c_3 would lead to solutions s_3 and s_4. If this were done, however, the new circle c_3 would have a negative radius, which is a geometric impossibility. Instead, we notice that if circle c_2 is reduced to a point and its radius added to that of c_1 and c_3, circles with centers at S_3 and S_4 are tangent to the red circles c_1 and c_3 while passing through C_2. Red c_1 intersects red c_3, but without affecting the validity of the reduction. Application of the two circle-one point construction will directly produce points S_3 and S_4 once the reduction described above has been made.

4. Two More Reductions. Similarily, Figure 11 shows how solutions S_5 and S_6 may be found by reducing circle c_1 to a point and adding its radius to c_2 and c_3. The last two solutions appear when circle c_3 is

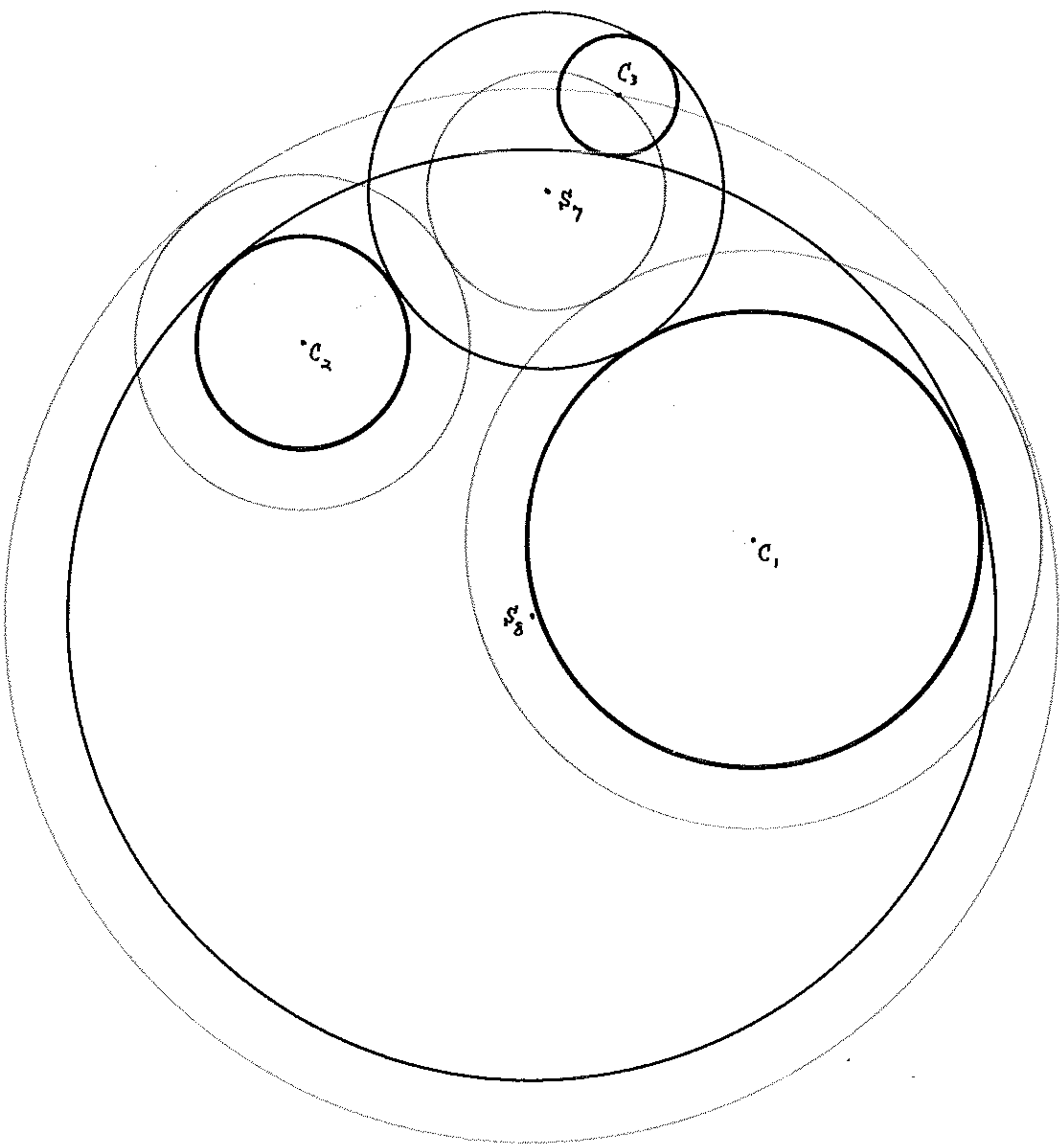


Figure 12. Solutions 7 and 8, r_3 Added.

reduced to a point, while this time adding the radius of circle c_3 to c_1 and c_2, as shown in Figure 12. Now that these reductions of the three circle problem to the two circle-one point problem have been found, the problem of Apollonius is essentially solved.

III. THE UNIFIED SOLUTION

A. First Steps.

After considerable effort, we have developed all the necessary constructions for solving the problem of Apollonius. These derivations appear perhaps complicated enough that it is difficult to separate the actual construction from the proof. Therefore a summary of the steps involved in solving the problem is in order. Obviously, we must work in reverse order from the way the constructions developed. The first step is to reduce the system of three given circles to an equivalent system of two circle and a point. This may be done in four ways, and all four ways must be used to get the eight possible cirlces tangent to the given three. One method of reduction is to subtract the radius of the smallest circle from the two larger ones, and reduce the smallest circle to the point at its center. The other three methods consist of reducing in turn each of the three given circles to its central point and adding to the radius of the other two circles the radius of the circle which is reduced. All four of these methods lead to a system of two circles and one point. The centers of the two circles which are tangent to the two reduced circles and passing through the center of the third circle are also the centers of the sought for solution circles. Figure 13 illustrates this and the following steps where c_2 is reduced to a point, and its

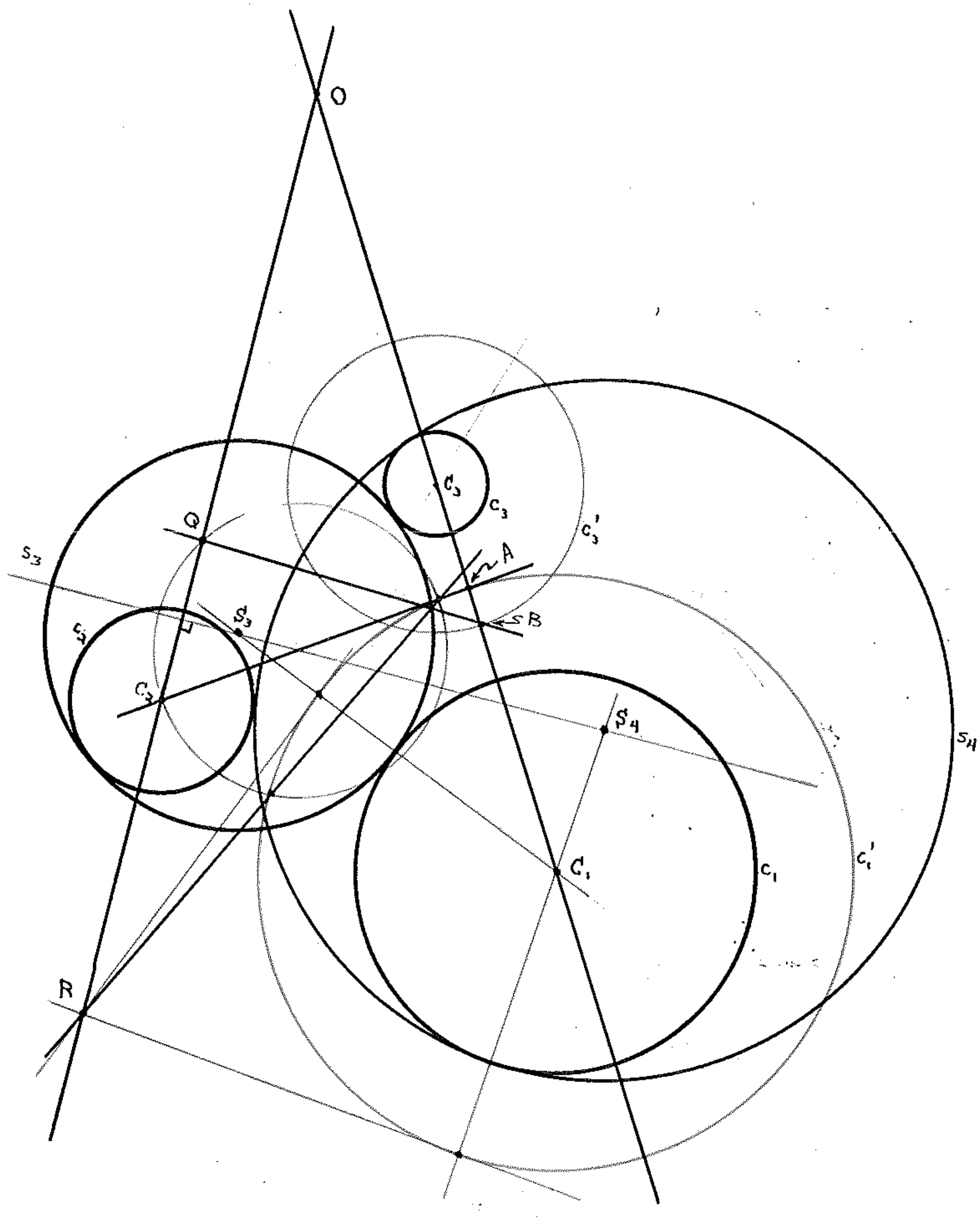


Figure 13. Construction for Solutions 3 and 4.

radius added to circles c_1 and c_3, producing circles c_1' and c_3'.

B. Further Reduction.

Now that one circle has been reduced to a point, the two circle-one point construction may be applied. Find the homothetic center O of the two reduced circles, c_1' and c_3' in Figure 13, by first constructing parallel radii in the two circles. Point O is then the intersection of the line through the end points of the two radii with the line of centers of the two circles. Note that O is not the homothetic center of circles c_1 and c_3. Next, draw the line from O to the center of the circle which has been reduced to a point, that is, C_2. Designating the points of intersection of circles c_1' and c_3' with line C_1C_3 points A and B, construct angle OBQ equal to angle OC_2A with point Q on line OC_2. The two circles passing through C_2 and Q and tangent to c_1' will also be tangent to c_3', and conversely. Thus the problem has been reduced to the one circle-two point case.

C. Completing the Solution.

We can choose either circle c_1' or c_3' to use with points C_2 and Q in order to find the required tangent circles. Let us use c_1' because it is larger than c_3', and constructions with larger figures are generally more accurate. Construct the perpendicular bisector of line segment QC_2, shown in red, and then pick some point on the bisector as the center of a circle passing through C_2 and Q while also intersecting circle c_1' in two points. The common chord of this arbitrary circle and circle c_1' meets line OQC_2 at a point R. Construct the two tangent lines from R to c_1', and draw the lines from the two points of tangency through C_1.

These two lines meet the perpendicular bisector of QC_2 at S_3 and S_4, the centers of the circles tangent to c_1' and passing through Q and C_2. The centers of the two circles tangent to c_1, c_2 and c_3 are also points S_3 and S_4. Merely draw circles with appropriate radii from S_3 and S_4 to complete the solution.

Only two of the eight solutions to the problem of Apollonius have been thus found, but the other six can be determined by an exactly similar method, except that circles c_1 and c_3 are reduced to points instead of c_2. Although this construction is theoretically accurate, in practice it is seldom very exact. Many shorter, and, therefore, practically more accurate constructions are known, but the one presented here is note-worthy for its straightforward motivation of reducing a complex problem to a simpler one. This solution to the problem of Apollonius demonstrates how the effective use of such basic geometric tools as reduction can solve advanced problems.

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2. Nathan Altshiller-Court, College Geometry (Richmond, Virginia: Johnson Publishing Co., 1925).

3. David Davis, Modern College Geometry (Cambridge, Mass.: Addison-Wesley Press Inc., 1949).

GLOSSARY

Homothetic Center. The two homothetic centers of a pair of circles are those points which divide the line of centers internally and externally in the ratio of the radii of the circles. We are concerned only with the external homothetic center (see Figure 7).

Power. The power of a point with respect to a circle is the product of a secant to the circle from the point and the external segment of that secant. When in the limiting case the secant becomes the tangent, the numerical value of the power of the point is easily seen to be the square of the length of the tangent. Of course, the power of a point is the same for all secants and tangents to the same circle from that point (see Figure 3).

Radical Axis. The radical axis of two circles is that line such that any point on it has equal powers with respect to both circles. As a corallary, the tangents from any point of the radical axis to both circles are equal. Thus when the two circles are tangent, the radical axis is the common tangent (see Figure 3). If the two circles intersect, the line joining the points of intersection, the common chord, is the radical axis (see Figure 4). If the two circlesdo not intersect or touch, their radical axis still exists and is perpendicular to the line of centers.

Radical Center. The radical center of three circles is that unique point which has the same power with respect to all three circles. It is determined by the intersection of the radical axes of the circles taken in pairs (Figure 2).

Secant. A secant is a line from a point to a circle which cuts the circle in two points. Its length is considered to be the distance from the exterior point to the further of the two points of intersection. Its external segment is that distance from the exterior point to the nearer point of intersection (Figure 3).